Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. (*) Let $n \in \mathbb{Z}^+$. We call a subset of $[n]$ odd if it has odd size, and even if it has even size.

Prove that the number of odd subsets of $[n]$ is equal to the number of even subsets of $[n]$.

2. (*) You have 20 different presents to distribute among 10 children. You may distribute presents in any way; it is even permissible that all the presents go to the same child. How many ways can the presents be distributed?

3. Repeat problem 2 but add the restriction that each child must get at least one present.

4. $n$ boys and $n$ girls all go to a dance. In how many ways can they partner up to dance together, assuming that each partnership is of one boy and one girl?

5. (*) Prove each of the following binomial identities in two ways; one using algebra, and one using combinatorics.

   a) $\binom{n}{2} + \binom{n+1}{2} = n^2$

   b) $\binom{n}{k} \binom{k}{j} = \binom{n}{n-j} \binom{n-j}{k-j}$

6. (*) You have 10 different colored balls in a single pile. You perform the following procedure:

   i. Split the balls into two piles (in any way)
   ii. Select a pile containing at least 2 balls (if it exists)
   iii. Split that pile into two piles (in any way)
   iv. Repeat steps ii & iii until you cannot

How many steps will you take until the procedure is completed? In how many different ways can this be done?

7. A 4-digit number is called a palindrome if it is the same when the digits are read in reverse. For example, 7337 and 3333 are 4-digit palindromes, but 1337 and 0990 are not. Note that 0990 doesn’t count because it’s actually a 3-digit number.

A 4-digit number is called an almost-palindrome if there is a way to change exactly one digit so that the result is a 4-digit palindrome. For example, 1337, 1501, and 1990 are 4-digit almost-palindromes (they could become
1331 or 7337, 1001 or 1551, and 1991), but 1234, 0991, and 1331 are not. The issue with 0991 is again that it is actually a 3-digit number, and the issue with 1331 is that if you change any digit, then it becomes a non-palindrome.

How many 4-digit almost-palindromes are there?