1 Exercises

1. Let \( a, b \) be any real numbers and
\[
(S) \begin{cases}
x + y = a \\
2x - 3y = b
\end{cases}
\]
Solve \((S)\). Rewrite \((S)\) using the column-by-column approach. Interpret the results.

2. In the following, determine whether \( W \) is a subspace of \( V \).

\( i ) \quad V = \mathbb{R}^3, W = \left\{ \begin{bmatrix} x \\ y \\ x + y + 1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \)

\( ii ) \quad V = \mathcal{M}_{nn}(\mathbb{R}), W \) is the subset of diagonal matrices, that is, matrices with non-zero entries only on the diagonal (the one from top-left to bottom-right).

3. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be linear and such that
\[
T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } T \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}
\]
Can you compute \( T \begin{bmatrix} a \\ b \end{bmatrix} \) for any \( a, b \in \mathbb{R} \) (justify your answer) ? If yes, do so.

4. Find two matrices \( A \) and \( B \) such that \((A + B)^2 \neq A^2 + 2AB + B^2\). When is this relation actually satisfied ?

5. Prove Theorem 1)\(v) \) from Chapter 5. That is, if \( A \) is any \( m \times n \) matrix then
\[
I_m A = A = AI_n
\]
2 Solution

1. One can reduce system \((R_2 \leftarrow R_2 - 2R_1)\) and see that it has only one solution

\[
\begin{align*}
x &= \frac{3a + b}{5} \\
y &= \frac{2a - b}{5}
\end{align*}
\]

Using the column-by-column approach, \((S)\) reads

\[
x \begin{bmatrix}
1 \\
2
\end{bmatrix} + y \begin{bmatrix}
1 \\
-3
\end{bmatrix} = \begin{bmatrix}
a \\
b
\end{bmatrix}
\]

We just saw that this problem has a unique solution no matter what \(a\) and \(b\) are : this means that \(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\) and \(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\) span \(\mathbb{R}^2\).

2. i) \(W\) is not a subspace since the 0-vector is not in it.

ii) \(W\) is a subspace : just observe that the zero-matrix is diagonal and that the sum of two diagonal matrices as well as any scalar multiple of a diagonal matrix stay diagonal\(^2\).

3. We saw in Exercise 1 that any \(\begin{bmatrix} a \\ b \end{bmatrix}\) can be computed as a (unique) linear combination of \(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\) and \(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\). By linearity, we can then compute any \(T \begin{bmatrix} a \\ b \end{bmatrix}\). We get

\[
T \begin{bmatrix} a \\ b \end{bmatrix} = T \left( \frac{3a + b}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2a - b}{5} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right) = \frac{3a + b}{5} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + \frac{2a - b}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}
\]

4. We can use

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

Observe that

\[
(A + B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}
\]

We can actually compute

\[
(A + B)^2 = A^2 + AB + BA + B^2
\]

\(^1\)And even more : not only they span it but there is a unique linear combination for each vector. We say that these two vectors form a basis of \(\mathbb{R}^2\) (see later in class).

\(^2\)Write it down : outside the diagonal one gets only \([A + B]_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0\) if \(i \neq j\) and \([\lambda A]_{ij} = \lambda a_{ij} = \lambda \times 0 = 0\) if \(i \neq j\).
so the equality holds if and only if \( AB + BA = 2AB \), i.e. iff

\[
AB = BA
\]

That is, when \( A \) and \( B \) commute. More generally, the binomial theorem holds for matrices if and only if they commute.

5. Let \( A \) be a \( m \times n \) matrix and let \( 1 \leq i \leq m, 1 \leq j \leq n \). We can compute the product \( I_mA \) which is a \( m \times n \) matrix as well and we have

\[
(I_mA)_{ij} = \sum_{k=1}^{m} (I_m)_{ik} A_{kj}
\]

But observe that \((I_m)_{ik} = 1\) if and only if \( i = k \) and is \( = 0 \) otherwise. Thus, only the term \( k = i \) remains in the sum, so

\[
(I_mA)_{ij} = A_{ij}
\]

This is valid for all \( 1 \leq i \leq m, 1 \leq j \leq n \) so this means that \( I_mA = A \).

The proof for \( AI_n \) is completely similar.