1. Consider a financial model that includes a stock, $S$ and a bank, and has two trading times \{0, 1\}. The bank will accept deposits or make loans at a one-period interest rate of $r = \frac{1}{4}$. An amount $B_0$ (positive for deposits, negative for loans) at time $t = 0$ will grow to $B_1 = (1 + r)B_0 = \frac{5}{4}B_0$ at time $t = 1$.

At time $t = 0$, any number of shares of stock can be bought or sold at the price $S_0 = $64. At time one, the stock price will be one of two values, $S_1(H) = $125 or $S_1(T) = $50

(a) Compute the initial prices $P^{50}_0$, $P^{100}_0$, and $P^{150}_0$ of put options on the stock with strike prices 50, 100, and 150 respectively.

(b) Compute the initial prices $C^{50}_0$, $C^{100}_0$ and $C^{150}_0$ of call options on the stock with strike prices 50, 100, and 150 respectively.

(c) Compute the initial price $C^{50}_0 - 2C^{100}_0 + C^{150}_0$ of the butterfly spread created by trading in calls. Also compute the initial price $P^{50}_0 - 2P^{100}_0 + P^{150}_0$ of the butterfly spread created by trading in puts. Is it the same or different from the initial price of the butterfly spread created using calls? Why?

2. [Exercise 1.14] (Another Alternative Definition of Arbitrage). Consider a financial model with two times $t = 0$ and $t = 1$. Assume that there is a bank at which one can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the one period interest rate $r \geq 0$, where $r$ is a constant that is known at time 0. Let us agree to say that a strategy is of type (Ar) provided that it is self-financing and the initial capital $X_0$ and terminal capital $X_1$ satisfy

(a) $X_1 \geq (1 + r)X_0$ for sure;

(b) There is a strictly positive probability that $X_1 > (1 + r)X_0$

Show that the model is arbitrage-free if and only if there are no strategies of type (Ar).

3. Consider a simple financial model with two times, $t = 0, 1$, a single stock, $S$, which pays no dividends, and a one period interest rate $r = .08$. The initial price per share of the stock is $S_0 = $40. Consider a contract that requires it’s owner to receive one share of stock in exchange for a payment of $K$ at time $t = 1$.

(a) What is the value of such a contract at $t = 0$ if $K = $45? If $K = $35?

(b) Find the payment $K$ that makes the value of the contract at $t = 0$ equal to zero.
4. Consider a financial model with two times, $t = 0$ and $t = 1$, and two stocks $S^1$ and $S^2$ that pay no dividends. We can buy or sell any number of shares of each of the stocks at $t = 0$ at the initial prices $S^1_0 = S^2_0 = $92. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest rate $r = .25$. There are three possible outcomes $\omega_1$, $\omega_2$ and $\omega_3$ regarding the stock prices, each having probability $\frac{1}{3}$. The possible stock prices at $t = 1$ are given by

$$S^1_1(\omega_1) = $210, \quad S^1_1(\omega_2) = $90, \quad S^1_1(\omega_3) = $60,$$
$$S^2_1(\omega_1) = $210, \quad S^2_1(\omega_2) = $180, \quad S^2_1(\omega_3) = $30.$$

Consider a derivative security $V$ with payoff at $t = 1$ given by

$$V_1(\omega_i) = \max\{S^1_1(\omega_i), S^2_1(\omega_i)\}, \quad i = 1, 2, 3,$$

i.e. if outcome $\omega_i$ occurs, the holder of the security receives the larger of $S^1_1(\omega_i)$ and $S^2_1(\omega_i)$ at $t = 1$. (This is an example of a basket option.) Let $V_0$ be the arbitrage-free price of $V$ at $t = 0$.

(a) Explain why we know that $92 < V_0 < \frac{210}{1.25}$ without finding a replicating strategy.

(b) Find a replicating strategy and use it to determine $V_0$.

5. Consider a simple financial market with three times $t = 0, 1, 2$ and a domestic currency, say dollars, and a foreign currency, say British pounds. In this model, we can

- Exchange any amount of dollars and pounds at $t = 0$ at the exchange rate $E_0^\£ = 1.5$, i.e. it costs $1.50 to purchase one pound at time 0.
- Borrow or invest any amount of dollars between $t = 0$ and $t = 1$ at the one-period interest rate $r_0^\$ = .08 and borrow or invest any amount of dollars between $t = 1$ and $t = 2$ at the one-period interest rate $r_1^\$ = .12. An amount $\alpha$ invested at $t = i$ will grow to the amount $\alpha(1 + r_i^\$)$ at $t = i + 1$. Similarly for loans. (In particular, an amount $\alpha$ invested at $t = 0$ and left in the bank until $t = 2$ will grow to $\alpha(1.08)(1.12)$ at $t = 2$.)
- Borrow or invest any amount of pounds between $t = 0$ and $t = 1$ at the one-period interest rate $r_0^\£ = .10$ and borrow or invest any amount of pounds between $t = 1$ and $t = 2$ at the one-period interest rate $r_1^\pounds = .15$.

Consider a contract made between two investors A and B at $t = 0$ in which it is agreed that Investor A will pay Investor B $2$ at each of the times $t = 1$ and $t = 2$ and Investor B will pay Investor A £2 at each of the times $t = 1$ and $t = 2$. Find the arbitrage-free price, in dollars, of Investor A’s position at $t = 0$. 

2