Due in class on Friday, January 29.

1. This problem refers to Example 1.23 in the text. In class, we used a replicating portfolio to compute the initial price of a put option with strike price \( K = 40 \) to be \( P_{0}^{40} = 6 \).

   (a) Compute the initial prices \( P_{0}^{30} \) and \( P_{0}^{50} \) of puts with strike prices 30 and 50 respectively.

   (b) Compute the initial prices \( C_{0}^{30} \), \( C_{0}^{40} \) and \( C_{0}^{50} \) of calls with strike prices 30, 40 and 50 respectively.

   (c) Compute the initial price \( C_{0}^{30} - 2C_{0}^{40} + C_{0}^{50} \) of the butterfly spread created by trading in calls.

   (d) Compute the initial price \( P_{0}^{30} - 2P_{0}^{40} + P_{0}^{50} \) of the butterfly spread created by trading in puts. Is it the same or different from the initial price of the butterfly spread created using calls? Why?

2. Consider a simple financial model with two times, \( t = 0, 1 \), a single stock, \( S \), which pays no dividends, and a one period interest rate \( r = .10 \). The initial price per share of the stock is \( S_{0} = 30 \). Consider a contract that requires its owner to receive one share of stock in exchange for a payment of \( K \) at time \( t = 1 \).

   (a) What is the value of such a contract at \( t = 0 \) if \( K = 35 \)? If \( K = 25 \)?

   (b) Find the payment \( K \) that makes the value of the contract at \( t = 0 \) equal to zero.

3. [Exercise 1.6] (Another Alternative Definition of Arbitrage). Consider a financial model with two times \( t = 0 \) and \( t = 1 \). Assume that there is a bank at which one can borrow or invest any amount of money between \( t = 0 \) and \( t = 1 \) at the one period interest rate \( r \geq 0 \), where \( r \) is a constant that is known at time 0. Let us agree to say that a strategy is of type \((Ar)\) provided that it is self-financing and the initial capital \( X_{0} \) and terminal capital \( X_{1} \) satisfy

   (a) \( X_{1} \geq (1 + r)X_{0} \) for sure;

   (b) There is a strictly positive probability that \( X_{1} > (1 + r)X_{0} \)

   Show that the model is arbitrage-free if and only if there are no strategies of type \((Ar)\).

4. [Exercise 1.12] Consider a financial model with two times, \( t = 0 \) and \( t = 1 \), and two stocks \( S_{1} \) and \( S_{2} \) that pay no dividends. We can buy or sell any number of shares of each of the stocks at \( t = 0 \) at the initial prices \( S_{0}^{1} = S_{0}^{2} = 20 \). There is also a bank at which we can borrow or invest any amount of money between \( t = 0 \) and \( t = 1 \) at the (one-period)
interest rate \( r = .1 \). There are three possible outcomes \( \omega_1, \omega_2 \) and \( \omega_3 \) regarding the stock prices, each having probability \( \frac{1}{3} \). The possible stock prices at \( t = 14 \) are given by
\[
S^1_1(\omega_1) = $24, \quad S^1_1(\omega_2) = $18, \quad S^1_1(\omega_3) = $16,
\]
\[
S^2_1(\omega_1) = $24, \quad S^2_1(\omega_2) = $24, \quad S^2_1(\omega_3) = $8.
\]
Consider a derivative security \( V \) with payoff at \( t = 1 \) given by
\[
V_i(\omega_i) = \max\{S^1_i(\omega_i), S^2_i(\omega_i)\}, \quad i = 1, 2, 3,
\]
i.e. if outcome \( \omega_i \) occurs, the holder of the security receives the larger of \( S^1_i(\omega_i) \) and \( S^2_i(\omega_i) \) at \( t = 1 \). (This is an example of a basket option.) Let \( V_0 \) be the arbitrage-free price of \( V \) at \( t = 0 \).

(a) Explain why we know that \( $20 < V_0 < \frac{$24}{1 + .1} \) without finding a replicating strategy.
(b) Find a replicating strategy and use it to determine \( V_0 \).

5. [Exercise1.16] Consider a simple financial market with three times \( t = 0, 1, 2 \) and a domestic currency, say dollars, and a foreign currency, say British pounds. In this model, we can

(a) Exchange any amount of dollars and pounds at \( t = 0 \) at the exchange rate \( E^P = 2 \), i.e. it costs $2 to purchase one pound at time 0.
(b) Borrow or invest any amount of dollars between \( t = 0 \) and \( t = 1 \) at the one-period interest rate \( r^d_0 = .1 \) and borrow or invest any amount of dollars between \( t = 1 \) and \( t = 2 \) at the one-period interest rate \( r^d_1 = .12 \). An amount \( \alpha \) invested at \( t = i \) will grow to the amount \( \alpha(1 + r^d_i) \) at \( t = i + 1 \). Similarly for loans. (In particular, an amount \( \alpha \) invested at \( t = 0 \) and left in the bank until \( t = 2 \) will grow to \( \alpha(1.1)(1.12) \) at \( t = 2 \).)
(c) Borrow or invest any amount of pounds between \( t = 0 \) and \( t = 1 \) at the one-period interest rate \( r^p_0 = .2 \) and borrow or invest any amount of pounds between \( t = 1 \) and \( t = 2 \) at the one-period interest rate \( r^p_1 = .15 \).

Consider a contract made between two investors A and B at \( t = 0 \) in which it is agreed that Investor A will pay Investor B $2 at each of the times \( t = 1 \) and \( t = 2 \) and Investor B will pay Investor A 2 pounds at each of the times \( t = 1 \) and \( t = 2 \). Find the arbitrage-free price, in pounds, of Investor B’s position at \( t = 0 \).