Exam #1 Review

Closed book and notes; calculators not permitted. Be sure to show all work and explain your reasoning as clearly as possible.

1. Consider the first order differential equation

\[
\frac{dy}{dt} = \frac{t}{2y + 1}
\]

(a) Find the general solution.
(b) Find the particular solution with \( y(1) = \frac{3}{2} \).
(c) Is this the unique solution with \( y(1) = \frac{3}{2} \)? How can you tell?

2. (a) Solve the initial value problem

\[
x y' + 3y = x^3, \quad y(1) = 10
\]

(b) What can you say about the existence and uniqueness of a solution to the above differential equation with initial condition \( y(3) = 2 \)? What about \( y(0) = 1 \)?

3. Sketch the phase line for the differential equation \( y' = (y - 1)(y + 1)(y + 2)^2 \) Label each critical point as stable, semi-stable or unstable.

4. Consider the differential equation

\[
\frac{dy}{dx} = (y - 1)(y + 2)^2(y^2 + 1).
\]

(a) Draw the phase line (phase diagram) for this differential equation.
(b) Let \( y(t) \) be the solution satisfying the initial condition \( y(0) = 0 \). Can the value of \( y(t) \) ever be less than \(-2\)? Why or why not?

5. Find the general solution for the differential equation

\[
\frac{dy}{dx} = \frac{x(y^2 + 1)}{2y}
\]
6. Consider the second order, linear differential equation

\[ y'' - 2y' - 8y = 0. \]

Find the general solution, and the particular solution satisfying \( y(0) = 0 \) and \( y'(0) = 0 \).

7. Consider the first order differential equation

\[ \frac{dy}{dt} + \sin(t)y = \sin(t) \]

(a) Find an integrating factor for the equation.
(b) Find the general solution.
(c) Find the particular solution with \( y(\frac{\pi}{2}) = 0 \).

8. Solve the initial value problem

\[ \frac{dy}{dt} = 3 - 2y \]
\[ y(0) = 0. \]

9. An impatient coffee drinker has placed his scalding hot (160°) coffee into a 40° refrigerator. After three minutes, the temperature has fallen to 140°. How long until the temperature of the coffee has fallen to a temperature that is safe to drink (120°). [You may want to use Newton’s Law of Cooling: \( \frac{dT}{dt} = -k(T - T^*) \). You may express your result in terms of natural logarithms of various numbers.]

10. A vat initially (at \( t = 0 \)) holds 100gal of pure water. A salt solution is added to the vat at a rate of 2gal/min. The concentration of this solution (in oz/gal) varies and is given by \( c(t) = \frac{1}{2}t(50 - t) \). The (well mixed) solution is allowed to flow out of the vat at a rate of 4gal/min. Find an expression for the amount of salt in the vat as a function of \( t \).