Exam #3 — Review

1. Let
   \[ A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ -2 & -4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \]

   (a) Determine the projection of \( b \) onto the column space of \( A \)?

2. Let
   \[ A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 4 & -2 \\ 2 & 1 & 8 & 0 \\ -2 & -1 & -8 & 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -1 \\ -3 \\ 7 \\ -1 \end{bmatrix} \]

   (a) Find an orthogonal basis for \( \text{row}(A) \).
   (b) Find a vector \( r \in \text{row}(A) \) and a vector \( n \in \text{nul}(A) \) such that \( v = r + n \). How many ways are there to do this?
   (c) What are the dimensions of \( \text{row}(A) \), \( \text{col}(A) \), and \( \text{nul}(A) \)? [Naturally, more than just a number is expected. How did you arrive at that number?]

3. Let \( A \) be an \( m \times n \) matrix. Justify the following claims:

   (a) For all vectors \( v \in \mathbb{R}^m \) there are vectors \( r \in \text{row}(A^T) \) and \( n \in \text{nul}(A^T) \) such that \( v = r + n \).
   (b) For every vector \( r \in \text{row}(A^T) \) there is a vector \( x \in \mathbb{R}^n \) such that \( r = Ax \).
   (c) For every vector \( c \in \text{col}(A^T) \) there is a vector \( x \in \mathbb{R}^n \) such that \( A^T Ax = c \).
   (d) For any vector \( b \in \mathbb{R}^m \), the equation \( A^T Ax = A^T b \) have at least one solution.

4. Consider the matrix
   \[ A := \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} \]

   (a) Find a basis for \( \text{nul}(A) \).
   (b) Find a basis for \( \text{col}(A) \).
   (c) Find the matrix for the orthogonal projection onto the nullspace of \( A \), \( \text{proj}_{\text{nul}(A)} \)

5. Let
   \[ \begin{cases} x_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} , x_2 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} , x_3 = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \end{cases} \]
(a) Apply the Gram-Schmidt process to the set \( \{x_1, x_2, x_3\} \) to obtain an orthogonal set \( \{v_1, v_2, v_3\} \).

(b) Let \( W = \text{span}(v_1, v_2) \). Find an orthogonal decomposition of the vector \( u = \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix} \) into \( u = w + w^\perp \) where \( w \in W \) and \( w^\perp \in W^\perp \).

6. (a) Is \( \{1 + x, 1 + x^2, x - x^2\} \) a basis for \( P_2 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\} \)?

(b) Show that for an \( m \times n \) matrix, \( A \), the solution set to \( Ax = b \) is a subspace of \( \mathbb{R}^n \) if and only if \( b = 0 \).