1. A scientist expects the quantities $x$ and $y$ to have the relationship

$$ax + by = 0$$

for some real numbers $a$ and $b$. Find the least squares values of $a$ and $b$ by using the data

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
<td>0</td>
</tr>
</tbody>
</table>

2. Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ -2 & -4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

(a) Find all possible least squares solutions for the system $Ax = b$.

(b) Determine the projection of $b$ onto the column space of $A$.

3. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 4 & -2 \\ 2 & 1 & 8 & 0 \\ -2 & -1 & -8 & 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -1 \\ -3 \\ 7 \\ -1 \end{bmatrix}$$

(a) Find an orthogonal basis for $\text{row}(A)$.

(b) Find a vector $r \in \text{row}(A)$ and a vector $n \in \text{null}(A)$ such that $v = c + n$. How many ways are there to do this?

(c) What are the dimensions of $\text{row}(A)$, $\text{col}(A)$, and $\text{null}(A)$? [Naturally, more than just a number is expected. How did you arrive at that number?]

4. Let $A$ be an $m \times n$ matrix. Justify the following claims:

(a) For all vectors $v \in \mathbb{R}^m$ there are vectors $r \in \text{row}(A^T)$ and $n \in \text{null}(A^T)$ such that $v = r + n$.

(b) For every vector $r \in \text{row}(A^T)$ there is a vector $x \in \mathbb{R}^n$ such that $r = Ax$.

(c) For every vector $c \in \text{col}(A^T)$ there is a vector $x \in \mathbb{R}^n$ such that $A^T Ax = c$.

(d) For any vector $b \in \mathbb{R}^m$, the normal equations $A^T Ax^* = A^T b$ have at least one solution $x^*$.  

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5. Let
\[ A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 7 \\ 7 \end{bmatrix} \]

(a) Compute \( A^T A \) and \( A^T b \).
(b) Find all least squares solutions \( x^* \) for the system \( Ax = b \).
(c) Find \( \text{proj}_{\text{col}(A)}(b) \)

6. Consider the matrix
\[ A := \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} \]

(a) Find a basis for \( \text{nul}(A) \).
(b) Find a basis for \( \text{col}(A) \).
(c) Find the matrix for the orthogonal projection onto the nullspace of \( A \), \( \text{proj}_{\text{nul}(A)} \)

7. Let
\[ \begin{align*} x_1 &= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, & x_2 &= \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, & x_3 &= \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \end{align*} \]

(a) Apply the Gram-Schmidt process to the set \( \{x_1, x_2, x_3\} \) to obtain an orthogonal set \( \{v_1, v_2, v_3\} \).
(b) Let \( W = \text{span}(v_1, v_2) \). Find an orthogonal decomposition of the vector \( u = \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix} \)

\[ \text{into} \quad u = w + w^\perp \quad \text{where} \quad w \in W \quad \text{and} \quad w^\perp \in W^\perp. \]

8. (a) Is \( \{1 + x, 1 + x^2, x - x^2\} \) a basis for \( \mathcal{P}_2 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\} \)?
(b) Show that for an \( m \times n \) matrix, \( A \), the solution set to \( Ax = b \) is a subspace of \( \mathbb{R}^n \) if and only if \( b = 0 \).