Exam #1 Review Problems

1. (a) Find all solutions to the system
\[
\begin{align*}
    x + 2y - 2z - 8w &= 0 \\
    2x + 4y - 3z - 13w &= 0 \\
    -x - 2y + 3z + 11w &= 0
\end{align*}
\]

Describe the nullspace of the matrix \( A = \begin{bmatrix} 1 & 2 & -2 & -8 \\ 2 & 4 & -3 & -13 \\ -1 & -2 & 3 & 11 \end{bmatrix} \) as a span of vectors.

(b) Is the vector \( b = \begin{bmatrix} -8 \\ -13 \\ 11 \end{bmatrix} \) in the column space of the matrix \( B = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -3 \\ -1 & -2 & 3 \end{bmatrix} \)? Why or why not?

2. (a) Find the \( LU \) factorization for the matrix
\[
M = \begin{bmatrix} -1 & 4 & 1 \\ 2 & -6 & 5 \\ -1 & 6 & 6 \end{bmatrix}
\]

and solve the system \( Mx = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix} \) as two triangular systems.

(b) Use Gauss-Jordan elimination to find the inverse of the matrix
\[
N = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}
\]

3. (a) Find a system of equations, with two equations and three unknowns, that has solutions
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.
\]

You may write the system in matrix form \( Ax = b \).

(b) Find a system, with three equations and three unknowns, having the same solutions as in part (a) and with \( b_3 = b_2 - b_1 \).

4. Find conditions on \( a, b, c, d, e, f, \) and \( g \) that ensure the matrix
\[
\begin{bmatrix}
    a & e & 0 & 0 \\
    a & b & f & 0 \\
    a & b & c & g \\
    a & b & c & d
\end{bmatrix}
\]
is invertible.