Math 880: Homework.

Assignment 1: Assigned Fri 09/06. Due Fri 09/20

1. Let $\sigma, \tau$ be two stopping times.
   (a) Show that $\sigma \land \tau, \sigma \lor \tau, \sigma + \tau$ are also stopping times.
   (b) If $\sigma \leq \tau$ almost surely, then show $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.
   (c) If $X$ is an integrable random variable, and $\sigma, \tau$ are two stopping times, show that $E(E(X | \mathcal{F}_\sigma) | \mathcal{F}_\tau) = E(X | \mathcal{F}_{\sigma \lor \tau})$.

2. (a) Let $\{F_n, n \in \mathbb{N}\}$ be a decreasing sequence of $\sigma$-algebras (i.e. $F_n \supseteq F_{n+1}$), and $\{X_n, F_n, n \in \mathbb{N}\}$ be a backward submartingale (i.e. $E(X_n | F_{n+1}) \geq X_{n+1}$). If $\inf_{n \in \mathbb{N}} EX_n > -\infty$, then show that $\{X_n, n \in \mathbb{N}\}$ is uniformly integrable.
   (b) Let $\{X_t, \mathcal{F}_t\}$ be a right continuous submartingale. Show that the function $t \mapsto EX_t$ is right continuous.

3. Let $\{X_t, \mathcal{F}_t \mid 0 \leq t < \infty\}$ be a right continuous martingale. Show that the following are equivalent.
   (a) $\{X_t \mid 0 \leq t \leq \infty\}$ is a uniformly integrable family of random variables.
   (b) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $X_t \rightarrow X_\infty$ in $L^1$ as $t \rightarrow \infty$. (Recall $\mathcal{F}_\infty = \sigma(\cup_t \mathcal{F}_t)$.)
   (c) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $X_t \rightarrow X_\infty$ almost surely and $\{X_t, \mathcal{F}_t \mid 0 \leq t \leq \infty\}$ is a martingale.
   (d) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $\{X_t, \mathcal{F}_t \mid 0 \leq t \leq \infty\}$ is a martingale.

4. (a) Let $X \overset{def}{=} \{X_t, \mathcal{F}_t \mid 0 \leq t < \infty\}$ be a right continuous supermartingale with $\sup_{t \geq 0} EX_t^+ < \infty$. We’ve shown in class $X_\infty \overset{def}{=} \lim_{t \rightarrow \infty} X_t$ exists almost surely and is integrable. Is $X_\infty$ a last element of $X$? Prove or find a counter example.
   (b) Let $X \overset{def}{=} \{X_t, \mathcal{F}_t \mid 0 \leq t < \infty\}$ be a non-negative, right continuous supermartingale. Show $X_\infty \overset{def}{=} \lim_{t \rightarrow \infty} X_t$ exists almost surely. Must $X_\infty$ be integrable? Is it a last element of $X$? Prove or find a counter example.

5. Let $X$ be a continuous local martingale. Show that there exists a localizing sequence $(\tau_n)$ such that for all $n$, the stopped process $X^{\tau_n} = \{X_{\tau_n \wedge t}, \mathcal{F}_t\}$ is bounded almost surely. (Note, we say $(\tau_n)$ is a localising sequence if it is an increasing sequence of stopping times which converges to $\infty$ almost surely, and for all $n$, the stopped process $X^{\tau_n}$ is a martingale).

Assignment 2: Assigned Mon 09/23. Due Mon 10/07

1. Let $M = \{M_t, \mathcal{F}_t \mid t \in [0, T]\}$ be a square integrable, continuous martingale with $M_0 = 0$ almost surely. Let $\langle M \rangle_T$ denote $\lim_{t \rightarrow T} \langle M \rangle_t$ (which exists since $\langle M \rangle_t$ is an increasing function of time). Show that
   $$P\left(\{\langle M \rangle_T < \infty\} \Delta \left\{\lim_{t \rightarrow T} M_t \text{ exists, and is finite} \right\}\right) = 0$$
   and
   $$P\left(\{\langle M \rangle_T = \infty\} \Delta \left\{\lim_{t \rightarrow T} |M_t| = \infty \right\}\right) = 0$$
   Here $A \Delta B = A \cup B - A \cap B$ denotes the symmetric difference between $A$ and $B$. [This is the moral equivalent of the statement $\lim_{t \rightarrow T} M_t$ exists and is finite "if and only if" $\lim_{t \rightarrow T} \langle M \rangle_t$ exists and is finite.]

2. (a) Let $\{\frac{1}{\sqrt{\lambda}} B_{\lambda t}\}_{\lambda > 0}$, for any $\lambda > 0$.
   (b) Let $\{B_{s+1} - B_s\}_{s \geq 0}$ for any fixed $s \geq 0$.

3. Let $W$ be a standard 1D Brownian motion. Do the following directly without appealing to the law of iterated logarithm, or the modulus of continuity theorems I quoted in class. (I will prove them later.)
   (a) For $\alpha < 1/2$, compute $\lim_{t \rightarrow 0^+} \sup \{\frac{\langle W \rangle_s - \langle W \rangle_t}{|t-s|^{1/2}} \mid 0 \leq s, t \leq 1 \land |t-s| < \delta\}$ if it exists (almost surely), and prove your answer.
   (b) Do the previous subpart for $\alpha > 1/2$.

4. Suppose $X$ is process with continuous trajectories and independent, stationary increments. (Recall, a process has stationary increments if the distribution of the increment $X_{t+h} - X_t$ only depends on $h$, and not on $t$.) Assume (for normalization) that $X_0 = 0 = EX_1$ and $EX_2^2 = 1$. Show that $X \in M_2$, and $\langle X \rangle_t = t$. [Thus if you know Lévy’s criterion, $X$ is a standard Brownian motion. Though not required for this problem, you may for simplicity assume $EX_1^2 < \infty$ if you like.]

5. For any $\lambda > 0$, we define the Poisson process with intensity $\lambda$ as follows. Let $\tau_1$ be a sequence of i.i.d exponential random variables with parameter $\lambda$ (i.e. $P(\tau_1 \in dt) = \frac{1}{\lambda} e^{-\lambda t} dt$). Let $\sigma_n = \sum_{i=1}^n \tau_i$. The intuition is that $\tau_i$ is the time at which the $i$th customer arrives, and $\sigma_n$ is the time it takes for the first $n$ customers to arrive. Now we define $N_t = \max\{n \in \mathbb{N} \mid \sigma_n \leq t\}$ to be the number of customers who’ve arrived up to time $t$. This is called the Poisson process with intensity $\lambda$.
   (a) Show that $N_t - \lambda t$ is an integer valued Poisson random variable with parameter $\lambda(t-s)$. Further, show $N_t - N_s$ is independent of $\mathcal{F}_s$. [Thus $N$ is a process with stationary, independent increments, that is not a Brownian motion. This is a counter example to the previous example, if the continuity in time assumption is dropped.]
   (b) Show that $\{(N_t - \lambda t)^2 - \lambda t\} \in M_2$. [This is the counter example to Lévy’s criterion, if the continuity in time assumption is dropped.]
Assignment 3: Assigned Mon 10/07. Due Fri 10/25

1. Let $M \in \mathcal{M}_c^2$ have absolutely continuous quadratic variation. Suppose $X \in \mathcal{L}(M)$, and $\tau$ is a stopping time. Let $\tilde{X}$ be defined by $\tilde{X}_t = \chi_{\{t \leq \tau\}} X_t$. Show that $\tilde{X} \in \mathcal{L}(M)$, $X \in \mathcal{L}(M^\tau)$, and $I(\tilde{X}, M) = I(X, M^\tau) = I(X, M)^\tau$. [We used this in class; if all of you are comfortable with this, then I’d recommend not discussing it further.]

2. Let $W$ be a standard 1D Brownian motion, and $\tau(x) = \inf\{t \mid x + W_t \not\in (0, \pi)\}$.
   (a) Compute $E\tau(x)$. $P(x + W_{\tau(x)} = 0)$ and $P(x + W_{\tau(x)} = \pi)$.
   (b) Define $u(x, t) = E\sin(x + W_{\tau(x) \wedge t})$. Find $u$ explicitly.

3. (Tanaka’s formula and local time) Let $W$ be a standard 1D Brownian motion, and define
   \[ L_t^\varepsilon = \frac{1}{2\varepsilon} \lambda \{ s \in [0, t] \mid |W_s| \leq \varepsilon \} \]
   where $\lambda$ denotes the Lebesgue measure. One naturally expects $\lim_{\varepsilon \to 0^+} L_t^\varepsilon$ measures the amount of time Brownian motion spends at 0. This problem proves the existence of limit.
   (a) Let $f_\varepsilon$ be the (unique) function such that $f_\varepsilon(0) = f_\varepsilon'(0) = 0$, and $f_\varepsilon'' = \frac{1}{\varepsilon^2} \chi_{[-\varepsilon, \varepsilon]}$.
   Show that
   \[ f_\varepsilon(W_t) - f_\varepsilon(0) = \int_0^t f_\varepsilon'(W_s) \, dW_s + L_t^\varepsilon \]
   (b) As $\varepsilon \to 0$, show that $E\int_0^1 |f_\varepsilon'(W_s) - \text{sign}(W_s)|^2 \, ds \to 0$.
   (c) As $\varepsilon \to 0$, show that $E|f_\varepsilon(W_t) - |W_t||^2 \to 0$.
   (d) Conclude there exists an adapted, square integrable process $L$ such that $E|L_t^\varepsilon - L_t|^2 \to 0$ as $\varepsilon \to 0$. Further show
   \[ |W_t| = \int_0^t \text{sign}(W_s) \, dW_s + L_t \]
   This is called Tanaka’s formula. [Remark: If we set $f(x) = |x|$, then $f'(x) = \text{sign}(x)$, and $f''(x) = 2\delta_0$. Thus if we formally apply Itô’s formula to $f(W)$, we exactly arrive at Tanaka’s formula. Of course, since $f \not\in C^2$, we may not apply Itô, and thus we have to resort to the approximations outlined above.]

4. (Bessel processes) Let $d > 1$, and $W$ be a standard $d$-dimensional Brownian motion. Let $R = |W|$.
   (a) Let $B = \sum_{i=1}^d \int_0^t \frac{W_s^{(i)}}{R_s} \, dW_s^{(i)}$. Show that $B$ is a standard 1D Brownian motion.
   (b) Show that $dR_t = \frac{d-1}{2R_t} \, dt + dB_t$. [Remark: Tanaka’s formula shows that for $d = 1$, this equation does not hold. The Bessel process can be used to study questions about the return of Brownian motion to the origin. We know for $d = 1$, $P(|W_t| > 0 \forall t > 0) = 0$. However, for $d > 1$, $P(|W_t| > 0 \forall t > 0) = 1$.
   It turns out that for $d = 2$, while Brownian motion does not return to the origin, it does come arbitrarily close. However, for dimensions 3 and higher, once Brownian motion leaves the origin, it remains bounded away from the origin almost surely. Details of the proof of this can be found in Karatzas and Shreve §3.3C]

5. Does there exist a process $Y$ adapted to $\{ X^{W}_t \}_{t \geq 0}$ with $E\int_0^1 Y_s^2 \, ds < \infty$ such that
   \[ \int_0^1 Y_s \, dW_s = \int_0^1 \left( W_s^2 - \frac{1}{2} \right) \, ds? \]
   If yes, find $Y$. If no, prove it.
Assignment 4: Assigned Wed 10/25. Due Wed 11/06

1. Let $\Omega = C[0,\infty)^d$, and $X$ be the canonical coordinate mapping process (i.e. $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$). For a fixed $t \geq 0$, define the time shift operator $\theta_t : \Omega \rightarrow \Omega$ by $\theta_t(\omega) = \omega(t+\cdot)$. That is $\theta_t \omega(s) = \omega(s+t)$. If $P$ is a probability measure on $\Omega$ under which $X$ is a Markov process (under the canonical filtration), then show that $P(\theta_t^{-1}F | F_t) = P(\theta_t^{-1}F | X_t)$.

2. Let $F$ denote the Borel $\sigma$-algebra on the Wiener space $C[0,\infty)^d$, and let $P$ be the Wiener measure. As we did in class, for any $x \in \mathbb{R}$, $F \in \mathcal{F}$, define $P^{P}(F) = P(F - x)$. Show that the function $x \mapsto P^{P}(F)$ is Borel measurable.

3. (a) Find a Markov process that is not a strong Markov process.
   (b) (Unrelated) Let $B$ be a Brownian motion. Find a Borel function $f$ such that the process $\{f(B_t)\}_{t \geq 0}$ is not a Markov process.

4. (a) Show that the process $X_t = |B_t|$ is a Markov process. Further show that the transition density $p_{\tau}(h,x,y) \overset{\text{def}}{=} P^P(X_{t+h} \in dy \; | \; X_t = x) = p(h,x,y) + p(h,x,-y)$, where $p$ is the transition density of Brownian motion, and $x,y \geq 0$.
   (b) Let $Y_t = M_t - B_t$, where $M_t = \sup_{s \leq t} B_s$. Show that $Y$ is a Markov process, with the same transition density as $X$. Conclude that $X$ and $Y$ have the same finite dimensional distributions.

5. Show that the augmented filtration of Brownian motion is right continuous.

6. (Harmonic functions) Let $\{W_t, \mathcal{F}_t\}$ and $\{P^x\}_{x \in \mathbb{R}^d}$ be Brownian family on $\mathbb{R}^d$, $D \subseteq \mathbb{R}^d$ be a bounded domain, and $\tau$ be the exit time of $W$ from $D$.
   (a) Show that $\tau < \infty$ almost surely.
   Given $f \in L^\infty(\partial D, \mathcal{B})$, we define $u(x) = E^x f(W_{\tau})$. [This is allowed since $\tau < \infty$ almost surely.]
   (b) Suppose $\sigma$ is a stopping time such that $\sigma \leq \tau$. Show that $u(x) = E^{x} u(W_\sigma)$.
   (c) (Mean value property) Suppose $B(x,R) \subseteq D$, and let $\sigma$ be the exit time of $W$ from $B(x,R)$. Show that
   \begin{equation}
   u(x) = \frac{1}{|\partial B(x,R)|} \int_{B(x,R)} u(y) d\sigma(y)
   \end{equation}
   where $d\sigma$ denotes the surface measure on $\partial B(x,R)$.

   It is a standard PDE result that once (1) is satisfied, $u \in C^2(D)$, and is harmonic (i.e. $\Delta u = 0$). However, without use of the mean value property, we can directly show that $u$ is harmonic provided we assume $u \in C^2(D)$.
   (d) If additionally $u \in C^2(D)$, show that $\Delta u = 0$.
   (e) Conversely, if $v \in C^2(D) \cap C(\overline{D})$ satisfies $\Delta v = 0$ in $D$, and $v = f$ on $\partial D$, show that $v(x) = E^x f(W_\tau)$.

Assignment 5: Assigned Wed 11/06. Due Wed 11/20

1. For any $b \in \mathbb{R}$, define $\sigma(b) = \inf\{s \mid W_s - s = b\}$. Compute $\lim_{b \to -\infty} \sigma(b)$.

2. (a) Let $Z_t = \exp\left(\int_0^t 2W_s dW_s - \frac{1}{2} \int_0^t 4W_s^2 ds\right)$. Verify that the Kazamaki and Novikov conditions will not directly show that $\{Z_t\}_{t \leq 1} \in \mathcal{M}_c[0,1]$. [This process is however a martingale, as we will see below.]
   (b) Suppose $(t_n) \nearrow \infty$ is sequence of times such that $E\exp\left(\frac{1}{2} \int_{t_n}^t |b_s|^2 ds\right) < \infty$. Show that $Z_t = \exp\left(\int_0^t b_s dW_s - \frac{1}{2} \int_0^t |b_s|^2 ds\right)$ is a martingale. [Feel free to use the Novikov condition.]
   (c) Suppose there exists a constant $C$ such that $|b_t(x)| \leq C(1 + |x|)$. Show that
   \[Z_t = \exp\left(\int_0^t b_s(W_s) dW_s - \frac{1}{2} \int_0^t |b_s(W_s)|^2 ds\right)\]
   is a martingale. Conclude that the process defined in part (a) is in $\mathcal{M}_c[0,1]$.

3. (Ornstein-Uhlenbeck process) Find an explicit solution of the SDE
   \[dX_t = \mu X_t dt + \sigma dW_t\]
   where $\mu, \sigma \in \mathbb{R}$, and $W$ is a 1D Wiener process. Also compute $EX_t$ and $\text{Var}(X_t)$.

4. (a) (Brownian bridge) Let $a, b \in \mathbb{R}$, $W$ be a 1D Wiener process. Show that strong existence and uniqueness holds for the SDE
   \[dX_t = \frac{b - X_t}{1-t} dt + dW_t; \quad t \in [0,1], X_0 = a.\]
   Show that $\lim_{t \to 1^-} X_t = b$ almost surely. [This is called a Brownian bridge from $a$ to $b$.]
   (b) For any $T \in (0,1]$, are the laws of $\{X_t\}_{t \leq T}$ and $\{W_t\}_{t \leq T}$ absolutely continuous? Prove it. Also, in the case the laws are absolutely continuous, find the Radon Nikodym derivative.

5. (Weak Uniqueness for Lipschitz coefficients) Let $b, \sigma$ satisfy the usual uniform Lipschitz and linear growth conditions. Suppose $\mu$ is a probability measure with finite variance. Show that any two (weak or strong) solutions to the SDE
   \[dX_t = b_t(X_t) dt + \sigma_t(X_t) dW_t\]
   with initial distribution $\mu$ have the same law. [Of course, the assumptions on $b, \sigma$ guarantee strong uniqueness, and a standard theorem (which we did not prove) guarantees that strong uniqueness implies weak uniqueness. While the general proof that strong uniqueness implies weak uniqueness is a little technical, the above can be done quickly with ‘bare hands’.]