# 1 Prove that \( f(x) = \sqrt{x} \) is uniformly continuous on \([0, \infty)\).
Step I: Notice that \( f(x) \) is uniformly continuous on \([0, 2] \) because \([0, 2] \) is compact. So given, \( \epsilon > 0 \), \( \exists \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) for \( |x - y| < \delta \).
Step II: Notice that \( f(x) \) is uniformly continuous on \([1, \infty)\). For \( |\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \).
Since \( x, y \in [1, \infty) \), we see that \( \sqrt{x} + \sqrt{y} \geq 2 \). So \( |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\frac{1}{2} \sqrt{x} + \frac{1}{2} \sqrt{y}} < \frac{2 \epsilon}{\delta} \). Thus for \( \epsilon > 0 \), take \( \delta = 2 \epsilon \).
The for \( |x - y| < 2 \epsilon, \frac{|x - y|}{\sqrt{x} + \sqrt{y}} < \frac{|x - y|}{2} < \frac{2 \epsilon}{2} = \epsilon \).
Step III: Thus for \( \epsilon > 0 \), let \( \delta = \min(1, 2 \epsilon, \delta_1) \). The for \( |x - y| < \delta \), we see that a) \( x \) and \( y \) are either both in \([0, 2] \) or both in \([1, \infty) \). b) If \( x, y \) are both in \([0, 2] \), then \( |x - y| < \delta < \delta_1 \) implies \( |f(x) - f(y)| < \epsilon \) by Step I. c) If \( x, y \) are both in \([1, \infty) \) then \( |x - y| < \delta < \frac{\delta}{2} \) implies \( |f(x) - f(y)| < \epsilon \) by Step II. Therefore uniformly continuous.

# 2 Let \( D \subset \mathbb{R} \). Let \( f : D \rightarrow \mathbb{R} \) be uniformly continuous on \( D \) and suppose \( \{x_n\} \) is a Cauchy sequence in \( D \). Then \( \{f(x_n)\} \) is a Cauchy sequence.
Pf: Given any \( \epsilon > 0 \), since \( f \) is uniformly continuous on \( D \) there exists a \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) whenever \( |x - y| < \delta \) and \( x, y \in D \). Since \( \{x_n\} \) is a Cauchy sequence, there exists a number \( N \) such that \( |x_n - x_m| < \delta \) whenever \( m, n > N \). Thus for \( m, n > N \) we have \( |f(x_n) - f(x_m)| < \epsilon \), so \( \{f(x_n)\} \) is a Cauchy sequence.

# 3 Let \( D \subset \mathbb{R} \). Let \( f : D \rightarrow \mathbb{R} \) be uniformly continuous on the bounded set \( D \). Prove that \( f \) is bounded on \( D \).
Pf: Suppose \( f(D) \) is not bounded.
Claim: There is a sequence \( s_n \in D \) such that \( f(s_n) \geq n, \forall n \). Pf: Construct the sequence by given \( n \in \mathbb{N} \). Define the set \( f(D, n) = \{x \in D : f(x) < n\} \). This nonempty because \( f(D) \) is not bounded. So choose \( s_n \) to be any point in \( f(D, n) \). Then choose \( s_{n+1} \) to be any point in \( f(D, n+1) \), etc. We get a sequence of points, \( s_n \) such that \( f(s_n) \geq n, \forall n \).
Now, since \( D \) is bounded we know that \( \{s_n\} \) has a convergent subsequence (call it \( \{s_{nk}\} \)). This subsequence is Cauchy (because all convergent sequences are). Then by the previous problem, \( f(\{s_{nk}\}) \) is Cauchy too. Thus \( f(\{s_{nk}\}) \) is convergent (because all Cauchy sequences are). But this is impossibly because \( f(\{s_{nk}\}) \geq n_k \).

# 4 Use the definition of derivative to find the derivative of \( f(x) = \sqrt{x} \) for \( x > 0 \).
\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.
\]

# 5 Let \( f(x) = x^2 \sin \left( \frac{1}{x^2} \right) \) for \( x \neq 0 \) and \( f(0) = 0 \).
a) Show that \( f \) is differentiable in \( \mathbb{R} \).
If \( x \neq 0 \) then \( f'(x) = 2x \sin \left( \frac{1}{x^2} \right) - 2x^2 \cos \left( \frac{1}{x^2} \right) \). If \( x = 0 \) then \( f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h^2 \sin \left( \frac{1}{h^2} \right) = \lim_{h \to 0} h \sin \left( \frac{1}{h^2} \right) = 0 \).
b) Show that \( f' \) is not bounded on the interval \([-1, 1]\).
Assume that \( f' \) is bounded. Then there exists \( M \) such that \( |f'(x)| \leq M \forall x \in [-1, 1] \). Then take \( x_0 \) such that \( x_0 > M \), \( x_0 > 1 \) and \( \sqrt{x_0} = n \) for some \( n \in \mathbb{N} \). Since \( M \) is finite, clearly \( x_0 \) exists. Look at \( \frac{1}{x_0} \). We see that \( \frac{1}{x_0} > 1 \), and \( |f'(\frac{1}{x_0})| = |\frac{2}{x_0} \sin(n\pi) - x_0 \cos(n\pi)| = x_0 > M \).