MA 355  Homework 1

• For each subset of \( \mathbb{R} \), give its maximum, minimum, supremum, and infimum, if they exist:
  \{1, 3\}: min and inf=1, max and sup=3
  \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}: min and inf=\(\frac{1}{2}\), max=none, sup=1
  (−\(\infty\), 4): min and inf=none, max=none, sup=4

p.21: #1. (a) Suppose \( r \in \mathbb{Q} \) and \( x \) is irrational and \( r + x \) is rational. Then \( r = \frac{p}{q} \) and \( r + x = \frac{s}{t} \) for \( p, q, s, t \in \mathbb{Z} \) where \( q, t \neq 0 \). Thus \( x = (r + x) - r = \frac{s}{t} - \frac{p}{q} = \frac{sq - pt}{qt} \). Since \( sq - pt \) and \( tq \) are integers and \( tq \neq 0 \), \( x \) is rational. This is a contradiction. ∴ If \( r \) is rational and \( x \) is irrational then \( r + x \) is irrational.

(b) (a) Suppose \( r \in \mathbb{Q} \) and \( x \) is irrational and \( rx \) is rational. Then \( r = \frac{p}{q} \) and \( rx = \frac{s}{t} \) for \( p, q, s, t \in \mathbb{Z} \) where \( q, t \neq 0 \). Thus \( x = rx \frac{1}{r} = \frac{s}{t} * \frac{p}{q} = \frac{sp}{tq} \). Since \( sp \) and \( tq \) are integers and \( tq \neq 0 \), \( x \) is rational. This is a contradiction. ∴ If \( r \) is rational and \( x \) is irrational then \( rx \) is irrational.

#2. Suppose there exists a rational number \( r \) whose square is 12. Write \( r \) in lowest terms, \( r = \frac{m}{n} \) where \( m, n \in \mathbb{Z} \) and have no factors in common. Then \( m^2 = 12n^2 = 3 \times 2^2 \times n^2 \). Since 3 appears in the right-hand side, 3 divides \( m^2 \). But 3 is prime, so it must divide \( m \); hence \( m^2 \) must be divisible by 9. But then 3 divides \( n^2 \) and therefore \( n \) as well, contrary to the assumption that \( m \) and \( n \) have no common factors. Thus there is no rational number whose square is 12.

# 4. Since \( E \) is nonempty, it has at least one element, say, \( x \). Since \( \alpha \) is a lower bound of \( E \), we know \( \alpha \leq x \). Similary, since \( \beta \) is an upper bound of \( E \), \( x \leq \beta \). By the transitivity of the order \( \leq \), we conclude \( \alpha \leq \beta \).

# 5. Since \( A \) is nonempty and bounded below, there exists \( \inf A \). By definition, \( \inf A \leq x, \forall x \in A \), so \( -\inf A \geq -x, \forall x \in A \), i.e., \( -\inf A \geq y, \forall y \in -A \). Thus \( -\inf A \) is an upper bound of \( -A \).

The set \( -A \) is thus nonempty and bounded above, hence there exists \( \sup(-A) \). Since \( \sup(-A) \geq y, \forall y \in -A, -\sup(-A) \leq -y, \forall y \in -A \), i.e., \( -\sup(-A) \leq x, \forall x \in A \). Thus \( -\sup(-A) \) is a lower bound of \( A \). Now \( -\inf A \geq \sup(-A), -\sup(-A) \leq \inf A \) since a lower bound is not bigger than the greatest lower bound and an upper bound is not smaller than the least upper bound. But these last inequalities are equivalent to \( -\inf A \geq \sup(-A), -\inf A \leq \sup(-A) \). Hence \( -\inf A = \sup(A) \).

• Suppose \( x, y \in \mathbb{R} \) and \( x < y \). Consider \( \tilde{x} = \frac{x}{\sqrt{2}} \) and \( \tilde{y} = \frac{y}{\sqrt{2}} \). By the density of the reals, there exists a rational number \( r \) such that \( \tilde{x} < r < \tilde{y} \) which implies \( \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \). Define \( w = r\sqrt{2} \) which is irrational by a previous #1b.