#1 Suppose that \( f(x) = x \) for all \( x \in [0, b] \). Show that \( f \) is integrable and that \( \int_a^b f(x)dx = \frac{b^2}{2} \).

Consider the partition \( P = \{0, \frac{b}{n}, \frac{2b}{n}, \ldots, \frac{(n-1)b}{n}, b\} \). So \( x_i = \frac{ib}{n} \) and \( \Delta x_i = \frac{b}{n} \). Since \( f \) is increasing we see that \( M_i(f) = \frac{ib}{n} \) and \( m_i(f) = \frac{(i-1)b}{n} \). Thus \( U(f, P) = \sum_{i=1}^{n} M_i(f) \Delta x_i = \sum_{i=1}^{n} \frac{ib}{n} \frac{b}{n} = \frac{b^2}{n^2} \sum_{i=1}^{n} \frac{n}{2} = \frac{b^2}{2} \). Similarly \( L(f, P) = \sum_{i=1}^{n} m_i(f) \Delta x_i = \sum_{i=1}^{n} \frac{(i-1)b}{n} \frac{b}{n} = \frac{b^2}{2} \sum_{i=1}^{n} (i-1) = \frac{b^2}{2} \frac{n(n-1)}{2} = \frac{b^2}{2} (\frac{n^2 - 1}{2}) \) so \( L(f) \geq \frac{b^2}{2} \). However, we know \( L(f) \leq U(f) \) so \( L(f) = U(f) \) which implies \( \int_a^b f = \frac{b^2}{2} \).

#2 Suppose \( f(c) = c \) for \( x \in [a, b] \). Show that \( f \) is integrable and that \( \int_a^b f(x)dx = c(b-a) \).

Similar to above but take \( P = \{a, a + \frac{(b-a)}{n}, \frac{2(b-a)}{n}, \ldots, a + \frac{(n-1)(b-a)}{n}, a + b - a\} \).

#3 Suppose \( f \geq 0 \), \( f \) is continuous on \([a, b] \) and \( \int_a^b f(x)dx = 0 \). Prove that \( f(x) = 0 \) for all \( x \in [a, b] \).

Proof by contradiction. Suppose \( f \geq 0 \), \( f \) is continuous on \([a, b] \) and \( \int_a^b f = 0 \). Also assume \( f(c) \neq 0 \) for some \( c \in [a, b] \). Then \( f(c) > 0 \). Since \( f \) is continuous we know there exists a \( \delta > 0 \) such that \( |f(x) - f(c)| < \frac{f(c)}{2} \) whenever \( |x - c| < \delta \). Let \( P \) be the partition of \([a, b] \) such that \( x_0 = a \), \( x_1 = \max(a,c - \delta) \), \( x_2 = \min(b,c + \delta) \) \( x_3 = b \). Then \( m_2 = \inf_{[x_1,x_3]} f \geq 0 \). Hence \( \int_a^b f \geq L(P, f) > 0 \). But this is a contradiction to \( \int_a^b f(x)dx = 0 \). Therefore \( f = 0 \) for all \( x \in [a, b] \).

#4 If \( f(x) \leq g(x) \) on \([a, b] \), then \( \int_a^b f(x)dx \leq \int_a^b g(x)dx \).

Let \( P \) be a partition of \([a, b] \) then \( L(f, P) \leq L(g, P) \). Since \( f \) is integrable we know \( \int_a^b f = \int_a^b f = \sup\{L(f, P)\} \leq \sup\{L(g, P)\} = \int_a^b g = \int_a^b f \).

#5 Suppose that \( f \) is integrable on \([a, b] \) and that there exists \( k > 0 \) such that \( f(x) \geq k \) for all \( x \in [a, b] \).

Prove that \( \frac{1}{f} \) is integrable on \([a, b] \).

Consider the function \( \frac{1}{f} \). Rewrite this as a composite function \( g = \frac{1}{f} \) and \( f = f(x) \) then \( g \circ f = \frac{1}{f} \). We know \( g \) is continuous on every point except \( 0 \) and since \( f(x) \geq k > 0 \) we know \( g \) is continuous on the range \([c, d] \) where \( f([a, b]) \subset [c, d] \). Since the composite of a continuous and integrable function is integrable, we find \( g \circ f \) is integrable on \([a, b] \) which means \( \frac{1}{f} \) is integrable on \([a, b] \).

#6 Prove the mean value theorem for integrals: If \( f \) is continuous on \([a, b] \), then there exists \( c \in (a, b) \) such that \( f(c) = \frac{1}{b-a} \int_a^b f \).

Suppose \( f \) is continuous on \([a, b] \). Let \( F(x) = \int_a^x f(t)dt \). By the fundamental theorem of calculus, \( F(x) \) is differentiable and \( F'(x) = f(x) \). Using the mean value theorem, we know there exists \( c \in (a, b) \) such that \( F'(c) = \frac{F(b) - F(a)}{b-a} \). Thus \( F'(c) = \int_a^b f(t)dt \). Therefore, \( \frac{1}{b-a} \int_a^b f(t)dt = f(c) \).

#7 Suppose \( f \) is a bounded real function on \([a, b] \), and \( f^2 \) is Riemann Integrable on \([a, b] \). Does it follow that \( f \) is integrable? Does the answer change if we assume \( f^3 \) is integrable?

Let

\[
 f = \begin{cases} 
 1 & x \text{ is rational} \\
 -1 & x \text{ is irrational} 
\end{cases}
\]

Then \( f \) is real and bounded, and \( f^2 \) is integrable but \( f \) itself is not integrable. Now suppose that \( f^3 \) is integrable. Consider the function \( \phi(t) = t^2 \). This function is defined and continuous on \( \mathbb{R} \). By the composition of functions theorem the function \( f = \phi(f^3) \) is therefore integrable.
#8 Let $f$ be continuous on $[a, b]$. Suppose that $\int_a^x f = \int_x^b f$ for all $x \in [a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Define $F = \int_a^x f - \int_x^b f = 0$. Then $F = \int_a^x f + \int_b^x f = 0$. By the Fundamental theorem of calculus we get $F' = f(x) + f(x) = 2f(x) = 0$. Therefore $f(x) = 0$. 