1. Let $K \in L^1(\mathbb{R}^n)$ and $a \in L^\infty(\mathbb{R}^n)$, and define the operator $T$ on $L^2(\mathbb{R}^n)$ by

$$(Ty)(x) = a(x) \int_{\mathbb{R}^n} K(x - y) g(y) \, dy.$$ 

If $|a(x)| \to 0$ as $|x| \to \infty$, prove $T$ is compact. (Hint: Operator-norm limits of compact operators are compact. Mollify $K$ and consider the case $a = 1_{|x| \leq R}$ first, using Arzela-Ascoli.)

2. (Evans 6.6.14, Courant minimax principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider an elliptic operator $L$ and its associated bilinear form $B$ defined by

$$Lu = - \sum_{j,k=1}^n \partial_{x_j} a_{jk}(x) \partial_{x_k} u, \quad B(u, v) = \int_{\Omega} Dv \cdot a(x) Du^T \, dx.$$ 

Assume the matrix $a(x)$ is symmetric, and $0 < c_0 \leq \xi^T a(x) \xi \leq c_1 < \infty$ for all $\xi \in S^{n-1}$ and $x \in \Omega$.

(a) Show that

$$\lambda_1 := \min_{u \in H^1_0(\Omega), \|u\|_{L^2} = 1} B(u, u)$$

is well-defined and any minimizer $u_1$ satisfies $Lu_1 = \lambda_1 u_1$ in the weak sense.

(b) Let $\Sigma_k$ denote the collection of $k$-dimensional subspaces of $H^1_0(\Omega)$, and for $S \in \Sigma_k$ let

$$S^\perp = \{ u \in H^1_0 : (u, v)_{L^2} = 0 \ \forall v \in S \}.$$ 

Prove that for each $k = 2, 3, \ldots$,

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{u \in S^\perp, \|u\|_{L^2} = 1} B(u, u)$$

is well-defined, and corresponds to an eigenvalue $Lu_k = \lambda_k u_k$ in the weak sense.

3. (Evans 7.5.6) Suppose $H$ is a Hilbert space and $u_k \rightharpoonup u$ in $L^2(0,T; H)$. Assume further that for some constant $C$, we have the uniform bounds $\text{ess sup}_{0 \leq t \leq T} \|u_k(t)\| \leq C$ (for $k = 1, 2, \ldots$). Prove that

$$\text{ess sup}_{0 \leq t \leq T} \|u(t)\| \leq C.$$ 

Note: We have $\int_a^b (v, u_k(t)) \, dt \leq C\|v\| \|b - a\|$ whenever $[a, b] \subset [0,T]$ and $v \in H$. 


4. Let \( \{e^{At}\}_{t \geq 0} \) be a \( C^0 \)-semigroup of operators in a Banach space \( X \), with infinitesimal generator \( A \). Let \( Z = C([0, T], X) \) and define the Duhamel map \( M: Z \to Z \) as follows: For \( f \in Z \), \( t \in [0, T] \),

\[
Mf(t) = \int_0^t e^{A(t-s)} f(s) \, ds.
\]

The semigroup is said to have \( (C^0) \) maximal regularity in \( X \) if for every \( f \in Z \) we have \( Mf(t) \in D(A) \) and \( Amf \in Z \). That is, the following map is in \( Z \) whenever \( f \in Z \):

\[
t \mapsto A \int_0^t e^{A(t-s)} f(s) \, ds.
\]

For \( p \geq 2 \), define a Banach space \( X_p \) of functions on \( \mathbb{R} \) as the completion of the Schwartz space \( S \) with respect to the norm

\[
\|u\|_p = \sup_{\xi \in \mathbb{R}} (1 + |\xi|^p) |\hat{u}(\xi)|.
\]

(Note: \( X_p \) consists of the Fourier transforms of all functions in a certain space of continuous functions, that are integrable on \( \mathbb{R} \).) Show that if \( A = \Delta \) is the Laplacian, so that \( v(t) = e^{At}u_0 \) is given by

\[
\hat{v}(t, \xi) = e^{-\xi^2t} \hat{u}_0(\xi),
\]

then \( \{e^{At}\} \) has the maximal regularity property in \( X_p \).

Remark: The \( C^0 \) maximal regularity property allows a beautiful abstract theory for nonlinear initial-value problems. But there are two very serious drawbacks to this theory: (i) A theorem of Baillon states that necessarily the Banach space \( X \) cannot be reflexive(!). (ii) Spaces like \( X_p \), defined above using the sup norm of Fourier transforms, are not invariant under smooth changes of variables (diffeomorphisms).