1. (Bressan’s model of morning traffic) Let \( \rho(x, t) \) model the density of cars on a one-lane highway \( (x > 0) \) with an entrance at \( x = 0 \) (and no exits...). We suppose that the velocity of cars \( v(\rho) = 1 - \rho \), and \( \rho(x, t) \) is a weak solution of the conservation law

\[
\rho_t + (\rho v(\rho))_x = 0, \quad x > 0, \ t > 0.
\]

(Note \( f(\rho) = \rho v(\rho) \) is concave.) At time \( t_0 = 0 \) (5am?) suppose the density is negligible: \( \rho(x, 0) = 0 \). During the morning commute, cars enter the road at a rate that first increases, then decreases. For simplicity suppose that \( \rho(t, 0) = t/2 \) \((0 < t < 1)\), and that after time \( t = 1 \), the entrance of cars is suddenly restricted (by ramp meters) so that \( \rho(t, 0) = 1/3 \) \((t > 1)\).

Describe a piecewise smooth weak solution of this initial-boundary value problem with admissible jumps (impinging characteristics). The solution should contain a (non-centered) rarefaction wave and a curved shock wave. Describe the ODE that determines the shock position, but you need not solve for the position explicitly as a function of time. Is the shock curve convex or concave? Draw the characteristic curves and interpret in terms of car velocity.

2. (Ballistic aggregation of shocks.) (a) For the Hopf equation (or inviscid Burgers equation)

\[
u_t + (\frac{1}{2}u^2)_x = 0 \quad (x \in \mathbb{R}, \ t > 1), \tag{1}
\]

show that a valid weak solution is one in which \( u(\cdot, t) \) is piecewise given by centered rarefaction waves, with \( u_x = 1/t \) everywhere except at a finite number of downward-jumping points of discontinuity \( x_1(t) < \ldots < x_n(t) \) which move at constant speeds \( x'_1(t), \ldots, x'_n(t) \) between times of collision.

(b) If we associate a ‘mass’ \( m_j \) and ‘momentum’ \( m_jv_j \) to the \( j \)-th shock according to

\[
m_j = t[u], \quad v_j = x'_j(t),
\]

then show that total mass and total momentum are conserved in collisions of two shocks.

3. A rather simple model for the evolution of cluster-size distributions \( n(x, t) \) is the Smoluchowski coagulation equation. In this model, \( \int_0^\infty n(y, t) \, dy \) represents the (expected) number of clusters whose size is in the interval \((0, x]\), \( x > 0 \). One seeks to model a process of coalescence of clusters of size \( x \) and size \( y \) into clusters of size \( x + y \), at rate \( K(x, y)n(x, t)n(y, t) \). In weak form, one requires that for any smooth test function \( \phi(x) \) with finite limit at \( \infty \),

\[
\partial_t \int_0^\infty \phi(x)n(x, t) \, dx = \frac{1}{2} \int_0^\infty \int_0^\infty (\phi(x + y) - \phi(x) - \phi(y))K(x, y)n(x, t)n(y, t) \, dx \, dy.
\]
Note (formally taking \( \phi(x) = x \)) that \( m_1 = \int_0^\infty x n(x,t) \, dx \) should be conserved in time.

(a) Suppose we normalize so \( m_1 = 1 \) and consider the case \( K(x,y) = x + y \). (This case turns out to exactly describe shock coalescence in certain random-walk solutions of the Hopf equation!) Taking \( \phi(x) = 1 - e^{-zx} \), show (formally) that

\[
    u(z,t) = \int_0^\infty (1 - e^{-zx}) n(x,t) \, dx
\]
satisfies \( u_t - uu_z = -u \) \( (z > 0) \).

(b) Given initial data \( u(z,0) = u_0(z) = \int_0^\infty (1 - e^{-zx}) n_0(x) \, dx \) for this PDE, with \( n_0(x) \) nonnegative of compact support and with \( m_1 = 1 \), show that an implicit formula for the solution \( u(z,t) \) is

\[
    u = e^{-t} u_0(z + (e^t - 1)w).
\]

Using the fact that \( u_0 \) is increasing and concave, with \( u_0(0) = 0, u_0'(0) = 1 \), show that this equation determines \( u(z,t) \) uniquely for all \( z > 0, t > 0 \).

4. (Level-set model of crystal growth or fire spreading) Suppose \( \gamma : \mathbb{R}^n \setminus \{0\} \to (0,\infty) \) is smooth and homogeneous. (This means \( \gamma(ap) = \gamma(p) \) for all \( a > 0 \); note that therefore \( 0 = p \cdot D\gamma(p) \).)

Consider the first-order PDE

\[
    \partial_t \phi + \gamma(\nabla \phi) |\nabla \phi| = 0, \tag{2}
\]

for \( \phi(x,t), x \in \mathbb{R}^n \setminus \{0\}, t > 0 \). Here the spatial gradient \( \nabla \phi = (\partial_{x_1} \phi, \ldots, \partial_{x_n} \phi) \).

(a) Suppose \( \phi \) is a smooth solution whose zero level set \( Z(t) = \{ x : \phi(x,t) = 0 \} \) is nondegenerate in the sense that \( \nabla \phi \neq 0 \) everywhere on \( Z(t) \). Suppose \( t \mapsto \hat{x}(t) \) is a smooth curve that lies in the zero level set of \( \phi \), so that \( \phi(\hat{x}(t),t) = 0 \) for all \( t \). Show that the normal velocity to \( Z(t) \) is given by \( \gamma \), i.e.,

\[
    \nu \cdot \frac{d\hat{x}}{dt} = \gamma(\nu),
\]

where \( \nu \) is a unit normal to \( Z(t) \) at \( \hat{x}(t) \).

(b) Observe that \( \phi(x,t) = (x - t \nu \gamma(\nu)) \cdot \nu \) is a solution for any fixed unit vector \( \nu \), with planar wave front \( Z(t) \). For fixed \( x \), the time the wave front reaches \( x \) is \( T(x,\nu) = x \cdot \nu / \gamma(\nu) \). Show that when \( \nu \) maximizes this as a function of \( \nu \) (with \( |\nu|^2 = 1 \)), we have

\[
    \frac{x}{T(x,\nu)} = D\gamma(\nu) + \gamma(\nu) \nu. \tag{3}
\]

Presuming the map \( (t,\nu) \mapsto t(D\gamma(\nu) + \gamma(\nu) \nu) \) from \((0,\infty) \times S^{n-1} \to \mathbb{R}^n \setminus \{0\} \) is bijective, the surface parametrized by \( \nu \mapsto D\gamma(\nu) + \gamma(\nu) \nu \) is called the Wulff shape for \( \gamma \).

(c) Solve the characteristic equations for the PDE (2), given initial data \( \phi(x,0) = \phi_0(x) = |x| - 1 \) (for which \( Z(0) \) is nondegenerate), and show that the projected characteristic starting at \( y \in S^{n-1} \) is on a curve \( (\hat{x}(y,t),t) \) that satisfies

\[
    \frac{\hat{x}(y,t) - y}{t} = D\gamma(\nu) + \gamma(\nu) \nu.
\]

where \( \nu \) is a unit normal to \( Z(0) \) at \( y \). Argue that as a consequence, the normalized surface \( Z(t)/t \) converges to the Wulff shape.
5. (A general unit-velocity level set motion model: \( \phi_t + |D\phi| = 0 \))

Let \( K \subset \mathbb{R}^n \) be a compact set, and let

\[
u(x,t) = \inf_{z \in K} |x - z| - t.
\]

(Then \( u(x,t) = 0 \) iff \( \text{dist}(x,K) = t \).) Prove that

(a) \(|u(x,t) - u(y,s)| \leq |x - y| + |t - s|\) for all \( x, y \in \mathbb{R}^n, t, s \geq 0 \).

(b) For any smooth \( v : \mathbb{R}^n \times (0, \infty) \to \mathbb{R} \) and any \((x_0,t_0) \in Q := (\mathbb{R}^n \setminus K) \times (0, \infty)\),

\((E^-)\) if \( u - v \) has a local max at \((x_0,t_0)\), then \( v_t + |Dv| \leq 0 \) at \((x_0,t_0)\).

\((E^+)\) if \( u - v \) has a local min at \((x_0,t_0)\), then \( v_t + |Dv| \geq 0 \) at \((x_0,t_0)\).

Thus \( u \) is a viscosity solution of \( u_t + |Du| = 0 \) in \( Q \) (except that \( u \) is not bounded).

Remark: The theory from class does not directly apply.