1. The parabolic equation

\[ u_t = (x^2u_x + u^2 - 2xu)_x \]

is related to a model for the energy distribution of photons in intergalactic space, known as the Kompaneets equation. Here \( x > 0 \) represents the magnitude of photon energy, and the diffusion term models the effect of small random changes in photon energy due to Compton scattering events with electrons.

For analysis of the equation above, a convenient discovery is that there is a “universal” supersolution having the form

\[ v(x,t) = x + \frac{1-x}{t} + g(t) \quad x \in (0,1), \quad t > 0. \]

Find a positive decreasing function \( g(t) \) with \( g(0^+) = \infty \) such that \( u \leq v \) for any nonnegative solution \( u \in C^{2,1}(U_T) \cap C^1(\bar{U}_T) \), which is defined for \((x,t)\) in a domain of the form

\[ U_T = (\epsilon, 1) \times (0, T), \]

and which satisfies the boundary conditions

\[ x^2u_x - 2xu = \begin{cases} 
0 & x = \epsilon, \ t > 0, \\
-u^2 & x = 1, \ t > 0.
\end{cases} \]

Suggestion: Look at \( w = v - u \) and consider the ‘first’ time when \( w = 0 \) for some \( x \in [\epsilon, 1] \). The cases \( x \in (\epsilon, 1) \) and \( x = \epsilon \) or \( x = 1 \) should be considered separately.

2. A few years ago I was asked to referee a paper in which the authors made the startling claim that the wave equation \( u_{tt} = \Delta u \) has smooth “superluminal” progressive wave solutions of the form

\[ u(x,t) = e^{i\omega t}v(x - ct), \quad c \in \mathbb{R}^3, \quad |c| > 1. \]

They were right! But they also insisted that there are such solutions \textit{with finite energy}

\[ E(t) = \int_{\mathbb{R}^3} |u_t|^2 + |Du|^2 \, dx \]

The paper was rejected. Show that no such solution is possible. (One approach: Consider the solution \( w(x,t) \) corresponding to “cut-off” initial data \( w(x,0) = \eta_\varepsilon(x)v(x) \)
where $\eta_\varepsilon$ is the standard mollifier and $\varepsilon$ is very large. Look at the energy of the two solutions and their difference in various balls.)

3. (compare Evans 2.5.22) Let $a > 0$ be a constant, and consider the telegraph equation

$$u_{tt} + au_t - u_{xx} = 0,$$

Prove that this equation has finite propagation speed: Suppose that $u$ is a $C^2$ solution, and $g(x) = h(x) = 0$ for $|x| > R$. Then $u(x, t) = 0$ for $|x| > R + t$. (I suggest you use an energy method, considering balls of the form $B(x, s)$ where $ds/dt = -1$.)

Side remarks: 1) If we let $v(x, t) = u(\varepsilon x, \varepsilon^2 t/a)/\varepsilon$ (“zooming out with a parabolic telescope”), then

$$\frac{\varepsilon^2}{a^2}v_{tt} + v_t - v_{xx} = 0,$$

Try sometime to prove that as $\varepsilon \to 0$, $v$ converges to a solution of the heat equation!


4. Suppose $u$ solves the wave equation

$$u_{tt} = \Delta u \quad \text{in } \mathbb{R}^3 \times (0, \infty),$$

$$u = g, \quad u_t = h \quad \text{on } \mathbb{R}^3 \times \{ t = 0 \},$$

where $g$ and $h$ are smooth with compact support. Using Kirchhoff’s formula, show that there exists a constant $C$ such that

$$\max_{x \in \mathbb{R}^3} |u(x, t)| \leq \frac{C}{t} \quad \text{for all } t > 0.$$

5. (Periodic wave forcing) Consider the initial value problem in $\mathbb{R}^3 \times \mathbb{R}$,

$$u_{tt} - \Delta u = q(x,t), \quad u(x, 0) = 0, \quad u_t(x, 0) = 0,$$

where $q$ is smooth with $q(x, t) = 0$ for $|x| \geq \rho > 0$. Show that if $q$ has the form $q(x, t) = e^{i\omega t}q_0(x)$, then there is a function $v(x)$ such that for each $x \in \mathbb{R}^3$,

$$u(x, t) - e^{i\omega t}v(x) \to 0 \quad \text{as } t \to \infty.$$