1. Let $K \in L^1(\mathbb{R}^n)$ and $a \in L^\infty(\mathbb{R}^n)$, and define the operator $T$ on $L^2(\mathbb{R}^n)$ by
\[(Tg)(x) = a(x) \int_{\mathbb{R}^n} K(x - y)g(y) \, dy.\]
If $|a(x)| \to 0$ as $|x| \to \infty$, prove $T$ is compact. (Hint: Operator-norm limits of compact operators are compact. Mollify $K$ and consider the case $a = 1_{|x| \leq R}$ first, using Arzela-Ascoli.)

2. (Evans 6.6.14, Courant minimax principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider an elliptic operator $L$ and its associated bilinear form $B$ defined by
\[Lu = -\sum_{j,k=1}^{n} \partial_{x_j} a_{jk}(x) \partial_{x_k} u, \quad B(u, v) = \int_{\Omega} Dv \cdot a(x) Du^T \, dx.\]
Assume the matrix $a(x)$ is symmetric, and $0 < c_0 \leq \xi^T a(x) \xi \leq c_1 < \infty$ for all $\xi \in \mathbb{S}^{n-1}$ and $x \in \Omega$.

(a) Show that
\[\lambda_1 := \min_{u \in H^1_0(\Omega), \|u\|_{L^2} = 1} B(u, u)\]
is well-defined and any minimizer $u_1$ satisfies $Lu_1 = \lambda_1 u_1$ in the weak sense.

(b) Let $\Sigma_k$ denote the collection of $k$-dimensional subspaces of $H^1_0(\Omega)$, and for $S \in \Sigma_k$ let
\[S^\perp = \{u \in H^1_0 : (u, v)_{L^2} = 0 \, \forall v \in S\}.\]
Prove that for each $k = 2, 3, \ldots$,
\[\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{u \in S^\perp, \|u\|_{L^2} = 1} B(u, u)\]
is well-defined, and corresponds to an eigenvalue $Lu_k = \lambda_k u_k$ in the weak sense.

3. (Evans 7.5.6) Suppose $H$ is a Hilbert space and $u_k \rightharpoonup u$ in $L^2(0, T; H)$. Assume further that for some constant $C$, we have the uniform bounds $\text{ess sup}_{0 \leq t \leq T} \|u_k(t)\| \leq C$ ($k = 1, 2, \ldots$). Prove that
\[\text{ess sup}_{0 \leq t \leq T} \|u(t)\| \leq C.\]
Note: We have $\int_a^b (v, u_k(t)) \, dt \leq C \|v\| |b - a|$ whenever $[a, b] \subset [0, T]$ and $v \in H$. 

4. (A nonlinear boundary-value problem) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^1$ boundary and let $a \in \mathbb{R}$. Write an appropriate weak formulation for the boundary value problem

$$-\Delta u + a \sin u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$ 

Using a contraction-mapping argument, prove that for $|a|$ sufficiently small, there is a unique solution $u \in H^1_0(\Omega)$ to the weak formulation.

(Remark: The map $u \mapsto \sin(u)$ is not Frechet differentiable on $L^2(\Omega)$. Indeed, if $g : \mathbb{R} \to \mathbb{R}$ is smooth and the composition map $u \mapsto g \circ u$ is Frechet differentiable on $L^2(\Omega)$, then $g$ must be affine!)