Problems due Friday Feb. 1

1. (a) Assume that \( f : \mathbb{R} \to \mathbb{R} \) is continuous, \( y : \mathbb{R} \to \mathbb{R} \) is \( C^1 \) and bounded, and \( y'(t) = f(y(t)) \) for all \( t \in \mathbb{R} \). Suppose \( y'(t_0) > 0 \) for some \( t_0 \in \mathbb{R} \). Prove \( y \) is increasing on \( \mathbb{R} \) (not necessarily strictly) and \( y_\infty := \lim_{t \to \infty} y(t) \) exists. (b) Prove also \( f(y_\infty) = 0 \).

2. (A one-sided uniqueness condition) Let \( (t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n \), \( a, b > 0 \), \( S = \{ y : |y - y_0| \leq b \} \), and \( R = \{ (t, y) : |t - t_0| \leq a, y \in S \} \). Assume \( f : \mathbb{R} \to \mathbb{R}^n \) is continuous and satisfies a “one-sided” Lipschitz condition for some \( L \geq 0 \), namely
\[
(f(t, \hat{y}) - f(t, y)) \cdot (\hat{y} - y) \leq L|\hat{y} - y|^2
\]
whenever \( t \in [t_0, t_0 + a] \) and \( \hat{y}, y \in S \). a) Prove that the initial value problem
\[
y'(t) = f(t, y(t)) , \quad y(t_0) = y_0 ,
\]
has at most one \( C^1 \) solution \( y : [t_0, t_0 + a] \to S \). (Follow the strategy: obtain an integral inequality and use Gronwall’s lemma.) b) Give an example with \( n = 1 \) where no solution exists exactly as described in a). c) Give an example with \( n = 1 \) of a non-Lipschitz function \( f \) independent of \( t \) such that the one-sided condition (1) holds.

3. A classic result in fluid dynamics by Yudovich from the 1960s (one that has hardly been improved upon since) concerns the uniqueness of fluid flow in the plane \( \mathbb{R}^2 \) with bounded vorticity. Yudovich showed that for \( (t, x) \in \mathbb{R} \times \mathbb{R}^2 \), the fluid velocity \( v(t, x) \in \mathbb{R}^2 \) satisfies a “log-Lipschitz” condition of the form
\[
|v(t, x) - v(t, \hat{x})| \leq Lr|\log r| , \quad \text{whenever } r = |x - \hat{x}| < \frac{1}{2} .
\]
Suppose now that particle paths \( x, \hat{x} : \mathbb{R} \to \mathbb{R}^2 \) move with the fluid, meaning
\[
x'(t) = v(t, x(t)) , \quad \hat{x}'(t) = v(t, \hat{x}(t))
\]
for all \( t \in \mathbb{R} \). With \( r(t) = |x(t) - \hat{x}(t)| \), it follows as in class that
\[
r(t) \leq r(t_0) + \int_{t_0}^{t} Lr(s)|\log r(s)| \, ds
\]
whenever \( t \geq t_0 \) and \( r(s) < \frac{1}{2} \) for all \( s \in [t_0, t] \). Show that if \( x(t) \) and \( \hat{x}(t) \) coincide for \( t = t_0 \), then they coincide for all \( t \geq t_0 \). (Remarks: 1. Please be careful to avoid division by zero! 2. A time-reversal argument yields coincidence for all \( t \leq t_0 \) as well.)

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4. Let $I \subset \mathbb{R}$ be an open interval, and suppose $f : I \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous, and globally Lipschitz in $y$, satisfying
\[
|f(t, \hat{y}) - f(t, y)| \leq L|\hat{y} - y|
\]
for all $t \in I$, $y, \hat{y} \in \mathbb{R}^m$. Let $t_0 \in I$, $\eta \in \mathbb{R}^m$, and consider the IVP
\[
y'(t) = f(t, y(t)) \quad (t \in I), \quad y(t_0) = \eta.
\]
Prove that Picard’s iteration method converges on the entire interval $I$ to a solution of the IVP, as follows: We let $y_0(t) = \eta$ for all $t \in I$, and inductively construct $y_n(t)$ for $n = 1, 2, \ldots$ from
\[
y_n(t) = \eta + \int_{t_0}^{t} f(s, y_{n-1}(s)) \, ds.
\]
Show (i) that $r_n(t) = |y_n(t) - y_{n-1}(t)|$ satisfies inequalities of the form $r_1(t) \leq M(t - t_0)$, $r_n(t) \leq \int_{t_0}^{t} K r_{n-1}(s) \, ds$ for $n \geq 2$, and infer $r_n(t) \leq M K^{n-1} \frac{(t - t_0)^n}{n!}$ whenever $t_0 \leq t \leq t_1 \in I$.

(The constants may depend upon $t_1$.) Then it remains to prove (ii) $y(t) = \eta + \sum_{n=1}^{\infty} (y_n(t) - y_{n-1}(t))$ exists and is continuous, (iii) $y(t)$ satisfies the IVP (integral version first), and (iv) provide a brief time-reversal argument to handle the case $t < t_0$.

5. Let $\alpha : [0, T] \to \mathbb{R}$ be an increasing function (not necessarily continuous) with $0 = \alpha(0) = \alpha(0^+)$ and let $f : [0, T] \to \mathbb{R}$ be continuous. For $0 \leq t \leq T$ and any partition $P = \{t_0, \ldots, t_n\}$ of $[0, t]$, with
\[
0 = t_0 < t_1 < \ldots < t_n = t, \quad \|P\| = \max_{j} |t_j - t_{j-1}|,
\]
note that the following Stieltjes integral exists:
\[
\int_{0}^{t} f(s) \, d\alpha(s) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j) (\alpha(t_j) - \alpha(t_{j-1})).
\]
Suppose $f$ satisfies
\[
0 \leq f(t) \leq \delta + \int_{0}^{t} f(s) \, d\alpha(s), \quad 0 \leq t \leq T.
\]
where $\delta \geq 0$. Prove that $f(t) \leq \delta e^{\alpha(t)}$ for $0 \leq t \leq T$.

(Suggestion: ‘Discretize’ the proof of Gronwall’s lemma, using a modulus of continuity for $f$, to obtain a bound on $f(t_j)$ using an inductive argument.)