Homework Assignment 8

1. (a) Let the permutation $\sigma \in S_5$ map $(1, 2, 3, 4, 5)$ to $(5, 4, 1, 2, 3)$. What does $\sigma^{-1}$ do to $(1, 2, 3, 4, 5)$? What about $\sigma^2$?

(b) Find the sign of $\sigma$, the sign of $\sigma^{-1}$, and the sign of $\sigma^2$.

(c) For any $n$, find the sign of the cyclic permutation satisfying $\sigma(j) = j + 1$ for $j = 1, \ldots, n - 1$, $\sigma(n) = 1$.

2. Prove that the determinant of an upper triangular matrix is the product of the diagonal entries. That is, if $A = (a_{ij})$ is an $n \times n$ matrix and $a_{ij} = 0$ whenever $i < j$, then show that
   \[
   \det A \overset{\text{def}}{=} \sum_{\sigma \in S_n} \text{sgn} \sigma \prod_{j=1}^{n} a_{\sigma(j), j} = \prod_{j=1}^{n} a_{jj}.
   \]

3. The aim of this problem is to show how the determinant of a linear transformation $T$ on a finite-dimensional vector space $V$ can be defined in an unambiguous way. Suppose $A = (v_1, \ldots, v_n)$ and $B = (w_1, \ldots, w_n)$ are two ordered bases for $V$. Then with respect to each basis, $T : V \to V$ is represented by matrices $A = M_{AA}(T)$, $B = M_{BB}(T)$.
   
   Prove that $\det A = \det B$. This means that any matrix representing $T$ in any basis has the same determinant, and we can define $\det T$ unambiguously to be this common value.

4. Use column reorderings and shears (adding a multiple of one column to another) to evaluate the determinant of the following companion matrix, and show that the $n \times n$ determinant
   \[
   \begin{vmatrix}
   x & -1 & 0 & \ldots & 0 \\
   0 & x & -1 & \ldots & 0 \\
   0 & 0 & x & \ldots & 0 \\
   \vdots & \ddots & \ddots & \ddots & \vdots \\
   c_0 & c_1 & c_2 & \ldots & (c_{n-1} + x)
   \end{vmatrix}
   = \pm \begin{vmatrix}
   -1 & 0 & 0 & \ldots & x \\
   x & -1 & 0 & \ldots & 0 \\
   0 & x & -1 & \ldots & 0 \\
   \vdots & \ddots & \ddots & \ddots & \vdots \\
   c_1 & c_2 & \ldots & (c_{n-1} + x) & c_0
   \end{vmatrix}
   = \pm p(x),
   \]
   
   where
   \[ p(x) = c_0 + c_1 x + \ldots + c_{n-1} x^{n-1} + x^n. \]
   
   Which signs above are correct? (The second matrix is obtained from the first by moving the first column to the last; every other column moves one place to the left. The original matrix has the form $xI + A$ where $a_{ij} = -1$ if $j = i + 1$ and $a_{nj} = c_{j-1}$, $j = 1, \ldots, n$.)