1. Prove that for any real \( p \geq 0 \),
\[
\left( \sum_{j=1}^{n} a_j b_j \right)^2 \leq \left( \sum_{j=1}^{n} a_j^p \right) \left( \sum_{j=1}^{n} b_j^{2p} \right)
\]
for all real numbers \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \). [The fact that \( \frac{1}{2} \) is sort of relevant, as you may suspect!]

2. Let \( V = P_3(\mathbb{R}) \) be the vector space of polynomials of degree 3 or less, with inner product given by
\[
(p, q) = \int_{-1}^{1} p(x)q(x) \frac{dx}{\sqrt{1-x^2}}
\]
Carry out the Gram-Schmidt orthogonalization procedure starting from the standard basis \( p_k(x) = x^k \), \( k = 0, 1, 2, 3 \), to find four orthonormal polynomials \( q_0, q_1, q_2, q_3 \), such that
\[
\text{span}\{q_0, \ldots, q_k\} = \text{span}\{p_k, \ldots, p_k\} \quad \text{for} \quad k = 0, 1, 2, 3.
\]
It may help to know that for \( k = 0, 2, 4, 6 \) we have
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{x^k}{\sqrt{1-x^2}} \, dx = 1, \frac{1}{2}, 3, \frac{5}{8}, \frac{15}{16}, \text{respectively.}
\]
You should get the normalized Chebyshev polynomials, which are featured prominently in Trefethen’s new book on approximation theory.

3. Given two vectors \( v \) and \( w \) in \( \mathbb{R}^n \), recall from problem 4.4 the definition of the wedge product \( v \wedge w \), whose components are indexed by pairs \((i, j)\) with \( 1 \leq i < j \leq n \). The norm of \( v \wedge w \) is its Euclidean norm as a vector in \( \mathbb{R}^{m_n} \) where \( m_n = \binom{n}{2} \): Thus
\[
\|v \wedge w\|^2 = \sum_{1 \leq i < j \leq n} |v_i w_j - v_j w_i|^2.
\]
(a) Prove the following sharp improvement of the Cauchy-Schwartz inequality:
\[
\|v\|^2 \|w\|^2 = (v, w)^2 + \|v \wedge w\|^2
\]
(Hint: What happens to the sum for \( \|v \wedge w\|^2 \) if you include terms with \( i = j \)?) All \( i \geq j \)?
(b) Infer from the previous part that for any linear map \( T : \mathbb{R}^n \to \mathbb{R}^n \) that is isometric (i.e., “orthogonal” we have
\[
\|(Tv) \wedge (Tw)\| = \|v \wedge w\|. 
\]

4. Suppose \( n \geq 2 \), and \( v, w \) are nonzero vectors in \( \mathbb{R}^n \). Prove that there is a linear isometry (i.e., an “orthogonal” map) \( T : \mathbb{R}^n \to \mathbb{R}^n \) such that \( Tv = \|v\| e_1 \) and \( Tw \in \text{span}\{e_1, e_2\} \), where \( e_j \) denotes the \( j^{\text{th}} \) standard unit basis vector.

5. A Pythagorean theorem for areas. Suppose \( n \geq 2 \), and \( v, w \) are nonzero vectors in \( \mathbb{R}^n \).
(a) With \( T \) as in the previous question, let \( \hat{v} = Tv = \|v\| e_1 \), \( \hat{w} = Tw \). Let \( \hat{P} \) be the parallelogram whose corners are at the points \( 0, \hat{v}, \hat{w}, \hat{v}+\hat{w} \). Let \( |\hat{P}| \) denote the area of \( \hat{P} \). Show \( |\hat{P}| = \|\hat{v} \wedge \hat{w}\| \).
(b) Whenever \( 1 \leq i < j \leq n \), let \( P_{ij} \) denote the parallelogram in \( \mathbb{R}^2 \) whose corners are the points
\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} v_i \\ w_j \end{pmatrix}, \begin{pmatrix} w_i \\ v_j \end{pmatrix}, \begin{pmatrix} v_i + w_i \\ v_j + w_j \end{pmatrix}.
\]
Show that this parallelogram has area \( |P_{ij}| = |v_i w_j - v_j w_i| \).
(c) Deduce that if \( P \) is the parallelogram in \( \mathbb{R}^n \) whose corners are the points \( 0, v, w, v+w \), then the area of \( P \) satisfies
\[
|P|^2 = \sum_{1 \leq i < j \leq n} |P_{ij}|^2.
\]
(You may assume \( |\hat{P}| = |\hat{P}| \).)