1. Determine whether \((1, 2, 3)\) belongs to the subspace of \(\mathbb{R}^3\) generated by \((3, 1, 0)\), \((4, 4, 2)\) and \((1, -1, -1)\). Explain your reasoning.

2. Let \(B = \{u_1, \ldots, u_m\}\) be a set of linearly independent vectors in a vector space \(V\), and let \(Y = \text{span} B\). Supposing \(w \notin Y\), let \(C = B + w = \{u_1 + w, \ldots, u_m + w\}\), and let \(Z = \text{span} C\). Show that \(C\) is a linearly independent set, and determine \(\dim Y \cap Z\).

3. Let \(V\) be a finite-dimensional vector space, and suppose \(X\), \(Y\), and \(Z\) are three subspaces of \(V\). Unfortunately, it is not generally true that

\[
\dim X + Y + Z = \dim X + \dim Y + \dim Z - \dim X \cap Y - \dim X \cap Z - \dim Y \cap Z + \dim X \cap Y \cap Z.
\]

Describe subspaces \(X\), \(Y\), \(Z\) that provide a counterexample to this formula for \(V = \mathbb{R}^3\). [Your example should have the property that \(X \cap (Y + Z) \neq (X \cap Y) + (X \cap Z)\).]

4. Let \(V = F^n\) where \(F\) is a field. Given any two vectors \(v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n)\) in \(V\), I wish to define the wedge product \(v \wedge w\) as a vector whose components consist of all \(2 \times 2\) determinants

\[
\det \begin{pmatrix} v_i & v_j \\ w_i & w_j \end{pmatrix} = v_i w_j - v_j w_i, \quad 1 \leq i < j \leq n.
\]

To make this a proper definition, we must enumerate and label all these components. Let \(I_n = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i < j \leq n\}\) and let \(m_n\) be the cardinality of \(I\). Let us fix some bijective map, written \((i, j) \mapsto i \wedge j\), that takes a pair \((i, j)\) in \(I_n\) to a unique integer \(i \wedge j\) in \(\{1, \ldots, m_n\}\). Then define \(v \wedge w\) as the vector in \(F^{m_n}\) whose \((i \wedge j)\)-th component \((v \wedge w)_{i \wedge j}\) is given by the determinant formula above.

(a) What is \(m_n\) explicitly as a function of \(n\)? Show that \(v \wedge w = -w \wedge v\).

(b) For \(v, w\) fixed, show that the maps \(u \mapsto u \wedge v\) and \(u \mapsto v \wedge u\) are both linear maps.

(c) If \(v \neq 0\), what is the kernel of the map \(u \mapsto v \wedge u\)? [Try the cases \(n = 2\) and \(n = 3\) first.]

[Preview: As we’re going to see, wedge products lead to a far-reaching generalization of the Pythagorean theorem, one that deals with \(k\)-dimensional area in \(\mathbb{R}^n\).]

5. Let \(F = \mathbb{Z}_2\), the field with only two elements. Suppose \(A\) is a set of \(k\) distinct prime numbers. Let us say that a natural number \(n > 0\) is \(A\)-philic if its factorization into primes (not necessarily distinct) contains only elements of \(A\). Prove that if \(B = \{n_1, \ldots, n_{k+1}\}\) is a set of \(k + 1\) distinct \(A\)-philic numbers, then there is a subset \(C \subset B\) such that the product of the numbers in \(C\) is a perfect square. [Hint: Associate to each \(n_j\) a vector in \(F^k\), and use facts about linear independence in \(F^k\).]