Problem (Pugh p.42 #12(c)): If \( x = A \mid B \) is a cut in \( \mathbb{Q} \), show that

\[
x = \text{l.u.b.} \{ r^* : r \in A \}.
\]

Solution: Let \( x = A \mid B \) be a cut in \( \mathbb{Q} \), and let \( S = \{ r^* : r \in A \} \). The proof involves two substeps. We must show \( x \) is an upper bound for \( S \) and that it is the least upper bound.

1. Let \( y \in S \). Then \( y = r^* \) where \( r \in A \). \( y \) is a cut and its left set is

\[
C = \{ q \in \mathbb{Q} : q < r \}.
\]

We have \( C \subseteq A \) since for any \( q \in \mathbb{Q} \), \( q < r \) implies \( q \in A \). Hence \( y \leq x \) by the definition of order on cuts. This proves \( x \) is an upper bound for \( S \).

2. Suppose \( z = C \mid D \in \mathbb{R} \) is any upper bound for \( S \). We claim \( x \leq z \), which means \( A \subseteq C \). Let \( r \in A \). Since \( A \) has no maximum element, there exists \( \hat{r} \in A \) with \( r < \hat{r} \). Since \( \hat{r}^* \in S \) we have \( \hat{r}^* \leq z \), and this means every rational number less than \( \hat{r} \) is contained in \( C \). Since \( r \) is such a rational number, \( r \in C \). Hence \( A \subseteq C \). This proves \( x \) is the least upper bound for \( S \).

Remark: Note that in the context of this problem, if \( a \in \mathbb{Q} \) and \( x \in \mathbb{R} \), “\( a \leq x \)” has no meaning, since rationals and cuts are different kinds of objects! “\( a^* \leq x \)” means \( \{ q \in \mathbb{Q} : q < a \} \subseteq A \), however.