Handout 1: Derivation of the Cartesian Equation for an Ellipse

The purpose of this handout is to illustrate how the usual Cartesian equation for an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is obtained from the Euclidean definition of the ellipse. Consider the ellipse shown in the following diagram\(^1\).

The Euclidean definition of the ellipse is that the total distance of the point \((x, y)\) from the two foci \((-c, 0)\) and \((c, 0)\) is equal to a constant. Following the notation suggested by the diagram we will label these distances \(d_1\) and \(d_2\), and the constant \(2a\), where \(a > 0\).

Now, using the Theorem of Pythagoras we can calculate that:

$$d_1 = \sqrt{(-c - x)^2 + y^2} = \sqrt{(c + x)^2 + y^2}$$

$$d_2 = \sqrt{(c - x)^2 + y^2} ,$$

\(^1\) This image was obtained from: [http://www.richland.edu](http://www.richland.edu)
bearing in mind that $x < 0$ in the diagram given above. The Euclidean condition for a point $(x, y)$ to be a point on the ellipse is then:

$$\sqrt{(c + x)^2 + y^2} + \sqrt{(c - x)^2 + y^2} = 2a.$$ 

Subtracting $d_2$ from both sides and squaring gives:

$$\sqrt{(c + x)^2 + y^2} = 2a - \sqrt{(c - x)^2 + y^2}$$

$$(c + x)^2 + y^2 = 4a^2 - 4a\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2.$$ 

Simplifying this equation and making the square root the subject gives:

$$4a\sqrt{(c - x)^2 + y^2} = 4a^2 + (c - x)^2 + y^2 - (c + x)^2 - y^2$$

$$4a\sqrt{(c - x)^2 + y^2} = 4a^2 - 4cx$$

$$\sqrt{(c - x)^2 + y^2} = a - \frac{c}{a} x.$$ 

Squaring both sides of this equation to remove the square root then gives:

$$(c - x)^2 + y^2 = a^2 - \frac{2c}{a} x + \frac{c^2}{a^2} x^2.$$ 

FOILing and simplifying this expression then gives:

$$c^2 + x^2 + y^2 = a^2 + \frac{c^2}{a^2} x^2.$$ 

Combining the $x^2$ terms, subtracting $c^2$ from both sides and simplifying gives:

$$\frac{a^2 - c^2}{a^2} x^2 + y^2 = a^2 - c^2.$$ 

Setting $b^2 = a^2 - c^2$ and dividing both sides of this equation by $b^2$ gives the familiar:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$