§7.4 #4 Minimize $\frac{1}{2}x^2 - 3xy + y^2 + \frac{1}{2}$ subject to the constraint $3x - y - 1 = 0$.

A: Use the method of Lagrange Multipliers. In this case we have

$$f(x, y) = \frac{1}{2}x^2 - 3xy + y^2 + \frac{1}{2}$$
$$g(x, y) = 3x - y - 1.$$

Now we need $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$, and $g(x, y) = 0$:

$$x - 3y = 3\lambda$$
$$-3x + 2y = -\lambda$$
$$3x - y - 1 = 0.$$

Solving the first two equations for $\lambda$ gives

$$\lambda = \frac{1}{3}x - y$$
$$\lambda = 3x - 2y,$$

and equating these gives

$$\frac{1}{3}x - y = 3x - 2y$$
$$8\frac{3}{3} = y.$$

Now plug this value into the third equation, which we will solve for $x$:

$$3x - 8\frac{3}{3} = 0$$
$$\frac{1}{3}x = 1$$
$$x = 3.$$
which means that
\[ y = \frac{8}{3}(3) = 8. \]
Therefore we have a minimum at \((3, 8)\), and the minimum value is \(f(3, 8) = \frac{1}{2}(3)^2 - 3(3)(8) + (8)^2 + \frac{1}{2} = -3.\)

§7.4 #6 Find the values of \(x, y\) that minimize the function
\[ x^2 + xy + y^2 - 2x - 5y, \]
subject to the constraint \(1 - x + y = 0.\)

A: Let \(f(x, y) = x^2 + xy + y^2 - 2x - 5y\) and let \(g(x, y) = 1 - x + y\). Then
\[
2x + y - 2 = -\lambda \\
x + 2y - 5 = \lambda \\
1 - x + y = 0.
\]
Solving the first two equations for \(\lambda\) gives
\[
\lambda = -2x - y + 2 \\
\lambda = x + 2y - 5.
\]
Now we can equate these and solve for \(x\) in terms of \(y\):
\[
-2x - y + 2 = x + 2y - 5 \\
3x = y + 7 \\
x = \frac{y + 7}{3}
\]
and plug this result into the constraint (i.e. the third equation):
\[
1 - \left(\frac{y + 7}{3}\right) + y = 0 \\
-\frac{4}{3} + \frac{2}{3}y = 0 \\
y = 2.
\]
Since \(y = 2\), \(x = \frac{2+7}{3} = 3.\) Therefore the minimum occurs at \((3, 2)\).

§7.4 #10 The amount of space required by a particular firm is \(f(x, y) = 1000\sqrt{6x^2 + y^2}\), where \(x\) and \(y\) are, respectively, the number of units of labor and capital utilized. Suppose that labor costs $480 per unit and capital costs $40 per unit and that the firm has $5000 to spend. Determine the amounts of labor and capital that should be utilized in order to minimize the amount of space required.
A: We are given that \( f(x, y) = 1000\sqrt{6x^2 + y^2} \). We need to find \( g(x, y) \). Recall that
\[
\text{Cost} = (\text{price}_x)(\text{quantity}_x) + (\text{price}_y)(\text{quantity}_y).
\]
So this means that, based on the information we are given, \( C(x, y) = 480x + 40y \), and so the constraint is \( C(x, y) = 5000 \). But notice that the constraint is not set equal to 0 like we are used to. We can easily fix this by rewriting \( C(x, y) \) as \( C(x, y) = 480x + 40y - 5000 \). So our three equations are
\[
\frac{6000x}{\sqrt{6x^2 + y^2}} = 480\lambda,
\frac{1000y}{\sqrt{6x^2 + y^2}} = 40\lambda,
480x + 40y - 5000 = 0.
\]
We can solve the first two equations for \( \lambda \) and we get
\[
\lambda = \frac{25x}{2\sqrt{6x^2 + y^2}},
\lambda = \frac{25y}{\sqrt{6x^2 + y^2}},
\]
which we equate and then solve for \( y \) in terms of \( x \):
\[
\frac{25x}{2\sqrt{6x^2 + y^2}} = \frac{25y}{\sqrt{6x^2 + y^2}}
\]
\[
\frac{25}{2}x = 25y
y = \frac{1}{2}x,
\]
and plug this result into the constraint:
\[
480x + 40\left(\frac{1}{2}x\right) - 5000 = 0
500x - 5000 = 0
500x = 5000
x = 10.
\]
Since \( x = 10, y = \frac{1}{2}(10) = 5 \). Therefore 10 units of labor and 5 units of capital should be utilized in order to minimize the amount of space required.

§7.4 #20 Find the values of \( x, y, z \) that minimize the function
\[
x^2 + y^2 + z^2 - 3x - 5y - z,
\]
subject to the constraint \( 20 - 2x - y - z = 0 \).
A: Here, \( f(x, y, z) = x^2 + y^2 + z^2 - 3x - 5y - z \) and \( g(x, y, z) = 20 - 2x - y - z \). Then
\[
\begin{align*}
2x - 3 &= -2\lambda \\
2y - 5 &= -\lambda \\
2z - 1 &= -\lambda \\
20 - 2x - y - z &= 0
\end{align*}
\]
Now we can take the first three equations, and solve each of them for \( x, y, \) and \( z \), respectively:
\[
\begin{align*}
2x &= 3 - 2\lambda \Rightarrow x = \frac{3}{2} - \lambda \\
2y &= 5 - \lambda \Rightarrow y = \frac{5}{2} - \frac{1}{2}\lambda \\
2z &= 1 - \lambda \Rightarrow z = \frac{1}{2} - \frac{1}{2}\lambda
\end{align*}
\]
Notice that all three of these equations are in terms of \( \lambda \) only. So we can just plug all three results into the constraint and solve the constraint for \( \lambda \):
\[
20 - 2\left(\frac{3}{2} - \lambda\right) - \left(\frac{5}{2} - \frac{1}{2}\lambda\right) - \left(\frac{1}{2} - \frac{1}{2}\lambda\right) = 0
\]
\[
20 - 3 + 2\lambda - \frac{5}{2} + \frac{1}{2} - \frac{1}{2} + \lambda = 0
\]
\[
14 + 3\lambda = 0
\]
\[
\lambda = -\frac{14}{3}
\]
Now we can plug this value into the equations we got for \( x, y, \) and \( z \):
\[
\begin{align*}
x &= \frac{3}{2} + \frac{14}{3} = \frac{37}{6} \\
y &= \frac{5}{2} + \frac{14}{6} = \frac{29}{6} \\
z &= \frac{1}{2} + \frac{14}{6} = \frac{17}{6}
\end{align*}
\]
So we have a minimum at \( (\frac{37}{6}, \frac{29}{6}, \frac{17}{6}) \).

§7.4 #24 A shelter for use at the beach has a back, two sides, and a top made of canvas. See Figure 4(b) on page 419. Find the dimensions that maximize the volume and require 96 square feet of canvas.

A: We know that the volume of the shelter is \( V(x, y, z) = xyz \). Now, the surface area of the canvas, based on the figure, is \( A(x, y, z) = xy + 2yz + xz - 96 \). So
\[
yz = \lambda y + \lambda z
\]
\[xz = \lambda x + 2\lambda z\]
\[xy = 2y\lambda + \lambda x\]
\[xy + 2yz + xz - 96 = 0.\]

Now we will solve the first two equations for \(\lambda\):
\[
\lambda = \frac{yz}{y + z}
\]
\[
\lambda = \frac{xz}{x + 2z}
\]

and equate them, solving for \(x\) in terms of \(y\):
\[
yz \left(\frac{xz}{x + 2z}\right) = xz(y + z)
\]
\[
yx + 2yz = yx + xz
\]
\[
x = 2y.
\]

Notice that all the \(z\)’s fell out. Now we need to equate the second and third equations:
\[
\lambda = \frac{xz}{x + 2z}
\]
\[
\lambda = \frac{xy}{x + 2y}
\]
\[
xz(x + 2y) = xy(x + 2z)
\]
\[
xz + 2yz = xy + 2yz
\]
\[
z = y.
\]

Now we can use these two results to substitute into the constraint, and solve for \(y\):
\[
(2y)x + 2y(y) + (2y)(y) - 96 = 0
\]
\[
6y^2 - 96 = 0
\]
\[
y^2 - 16 = 0
\]
\[
y = \pm 4.
\]

Because we cannot have negative dimensions, we throw out the solution \(y = -4\). So \(y = 4\). Then \(x = 2(4) = 8\), and \(z = 4\). So the volume is maximized when the dimensions are \((8, 4, 4)\).