All pairs shortest path problem.

Given a matrix $D = ||b_{ij}||$ of edge lengths

Find a shortest path between every pair of vertices - assume no negative cycles. $\forall v$ vertices
Floyd's Algorithm

Initialise: \( d_{ij} = c_{ij} \quad \forall i, j \)

For \( k = 1 \) to \( n \) do

For \( i = 1 \) to \( n \) do

For \( j = 1 \) to \( n \) do

\[ d_{ij} = \min \{ d_{ij}, d_{ik} + d_{kj} \} \]

Claim: after \( k \) outer loop steps, \( d_{ij} \) is the minimum length of a walk (path) from \( i \) to \( j \), all of whose interior vertices are in \( \{1, 2, \ldots, k\} \).
Proof of Claim: by induction on \( k \).

\( k = 0 \): \( d_{ij} = \text{length of edge } (i,i) \)

Suppose true for some \( k \geq 0 \).

Consider the walks from \( i \) to \( j \) that use \( e_1, e_2, \ldots, e_{k+1} \) inside.

(a) Walks that do not use \( e_{k+1} \)  
(b) Walks that use \( e_{k+1} \)  

Min length = \( d_{ij} \)
Assignment Problem

$n$ people & $n$ tasks:

$C_{ij}$ = cost of assigning person $i$ to task $j$.

Problem: assign person $i$ to task $T(i)$ s.t.

1) Each task has one assigned person
2) The sum $\sum_i C_{ij} n(i)$ is minimized
Suppose we have solved the $k \times k$ problem.

Given this solution, I try to reduce my solution to the $(k+1) \times (k+1)$ problem to a shortest path problem.
Let $M'$ be any $(k+1) \times (k+1)$ matching
Look at $M \oplus M' = (M \backslash M') \cup (M' \backslash M)$

- plus
- plus more cycles

Removing

only vertices degree 1 in $M \backslash M'$; others have degree 0 or 2
Orient all edges: M from right to left; Nst from left to right

\[
\text{length} = \text{cost}
\]

\[M@M' = P - \{e_{x,y} \in E \text{ plus cycle } C_1, C_2\}\]

\[\text{cost}(M') - \text{cost}(M) = l(P) + \sum_i l(C_i)\]

So we should just minimize \(l(P)\).

If \(l(C) < 0\) \(\Rightarrow\) M not optimal.