Department of Mathematical Sciences
Carnegie Mellon University

Operations Research II 21-393

Answers to homework 2.

Q1 Solve the following 2-person zero-sum games:

\[
\begin{bmatrix}
6 & 2 & 4 \\
5 & 2 & 5 \\
4 & 1 & -3 \\
\end{bmatrix}
\begin{bmatrix}
2 & 1 & 1 & 0 & -1 \\
4 & 3 & 2 & 1 & -1 \\
1 & 1 & 0 & -1 & 1 \\
2 & 1 & 1 & -2 & -2 \\
4 & 1 & 0 & -2 & -3 \\
\end{bmatrix}
\]

Solution (2,2) is a saddle point for the first game. Thus the solution is for player A to use 1 and player B to use 2. The value of the game is 2.

For the second game we have the following sequence of row/column removals because of domination:

Remove column strategy 1.

\[
\begin{bmatrix}
1 & 1 & 0 & -1 \\
3 & 2 & 1 & -1 \\
1 & 0 & -1 & 1 \\
1 & 1 & -2 & -2 \\
1 & 0 & -2 & -3 \\
\end{bmatrix}
\]

Remove column strategy 2.

\[
\begin{bmatrix}
1 & 0 & -1 \\
2 & 1 & -1 \\
0 & -1 & 1 \\
1 & -2 & -2 \\
0 & -2 & -3 \\
\end{bmatrix}
\]

Remove column strategy 3.

\[
\begin{bmatrix}
0 & -1 \\
1 & -1 \\
-1 & 1 \\
-2 & -2 \\
-2 & -3 \\
\end{bmatrix}
\]

Remove row strategy 1.

\[
\begin{bmatrix}
1 & -1 \\
-1 & 1 \\
-2 & -2 \\
-2 & -3 \\
\end{bmatrix}
\]
Remove row strategy 4.  \[
\begin{bmatrix}
1 & -1 \\
-1 & 1 \\
-2 & -3
\end{bmatrix}
\]

Remove row strategy 5.  \[
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]

The optimal strategies for this game are for player A to play rows 2 and 3 with probability 1/2 each. Similarly, player B plays columns 4 and 5 with probability 1/2 each.

**Q2:** Suppose the \( n \times n \) matrix \( A \) is such that all row and column sums are equal to the same value \( C \). What is the solution to this game?

**Solution** We can assume that \( C > 0 \). We know that

\[
P_A^{-1} = \min x_1 + x_2 + \cdots + x_n : A^T \mathbf{x} \geq 1, \; \mathbf{x} \geq 0.
\]

\[
P_B^{-1} = \min y_1 + y_2 + \cdots + y_n : A \mathbf{y} \geq 1, \; \mathbf{y} \geq 0.
\]

Next observe that \( x_i = y_j = C^{-1} \) for all \( i, j \) gives feasible solutions to these dual problems with the same value \( nC^{-1} \). It follows that \( P_A = P_B = C/n \) and that the optimal strategy for each player is to play each strategy with probability \( 1/n \).

**Q3** The correlation coefficient between assets A and B is .1 and the other data is given below:

<table>
<thead>
<tr>
<th>ASSET</th>
<th>( \bar{r} )</th>
<th>( \sigma )</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>.1</td>
<td>.15</td>
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<tr>
<td>B</td>
<td>.18</td>
<td>.3</td>
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</tbody>
</table>

(a) Find the proportions \( \alpha \) of A and \( 1 - \alpha \) of B that define a portfolio having minimum standard deviation.

(b) What is the value of this minimum standard deviation.

(c) What is the expected return for this portfolio.

**Solution** (a) Let \( \sigma(\alpha) \) be the standard deviation of \( \alpha A + (1 - \alpha) B \). Then

\[
\sigma(\alpha)^2 = .0225\alpha^2 + .009\alpha(1 - \alpha) + .09(1 - \alpha)^2.
\]

This is minimized when

\[
.045\alpha + .009 - .018\alpha - .18(1 - \alpha) = 0.
\]
or
\[ \alpha = \frac{171}{207}. \]
(b)
\[ \sigma(171/207) \approx 0.149. \]
(c) The expected return is
\[ 0.1 \times \frac{171}{207} + 0.18 \times \frac{36}{207} = \frac{2378}{20700}. \]

**Q4** Suppose there are \( n \) assets which are uncorrelated. The mean return \( \bar{r} \) is the same for each asset. The return on asset \( i \) has a variance of \( \sigma_i^2 \).

(a) Describe the efficient set.

(b) Find the minim-variance point.

**Solution** (a) Since each asset has the same average return \( \bar{r} \), each portfolio will have the same average return \( \bar{r} \) and so the efficient set consists of a single point, \((\sigma, \bar{r})\). It remains to compute \( \sigma \).

(b) Since \( \sum_{i=1}^{n} w_i \bar{r} = \bar{r} \) is implied by \( \sum_{i=1}^{n} w_i = 1 \) we can drop one constraint in our Lagrangean formulation. Our equations then become
\[ \sigma_i^2 w_i = \mu \quad i = 1, 2, \ldots, n. \]

\( w_1 + \cdots + w_n = 1 \) then implies that
\[ \mu = \left( \sum_{j=1}^{n} \sigma_j^{-2} \right)^{-1} \]
and
\[ w_i = \sigma_i^{-2} \left( \sum_{j=1}^{n} \sigma_j^{-2} \right)^{-1} \]
and
\[ \sigma^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 = \left( \sum_{j=1}^{n} \sigma_j^{-2} \right)^{-1}. \]

**Q5** There are 3 assets with data given below:
\[ V = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \bar{r} = \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} \]
(a) Find the minimum variance portfolio.

(b) Find another efficient portfolio by setting $\lambda = 1, \mu = 0$.

(c) If the risk-free rate is $r_f = .2$, find the efficient portfolio of risky assets.

**Solution** The equations determining efficient portfolios are

$$
2w_1 + w_2 + \cdot4\lambda + \mu = 0
$$

$$
w_1 + 2w_2 + w_3 + \cdot8\lambda + \mu = 0
$$

$$
w_2 + 2w_3 + \cdot8\lambda + \mu = 0
$$

$$
w_1 + w_2 + w_3 = 1
$$

$$
.4w_1 + .8w_2 + .8w_3 = r
$$

where $r$ is the target return.

(a) $r$ is not specified i.e. there is no requirement. Thus we drop the 5th equation and put $\lambda = 0$.

By symmetry $w_1 = w_3 = w$ and the equations become

$$
2w + w_2 + \mu = 0
$$

$$
2w + 2w_2 + \mu = 0
$$

$$
2w + w_2 = 1
$$

So

$$
w_1 = \frac{1}{2}, w_2 = 0, w_3 = \frac{1}{2}, \lambda = 0, \mu = -1
$$

is the solution.

(b) Now the equations become

$$
2w_1 + w_2 + \cdot4 = 0
$$

$$
w_1 + 2w_2 + w_3 + \cdot8 = 0
$$

$$
w_2 + 2w_3 + \cdot8 = 0
$$

So

$$
w_1 = -1, w_2 = -2, w_3 = -3, \lambda = 1, \mu = 0.
$$

To get the actual solution we scale this to give

$$
w_1 = 1/6, w_2 = 1/3, w_3 = 1/2.
$$

(c) Following the argument in Section 6.9, we solve the following equations (6.10):

$$
\sum_{i=1}^{n} \sigma_{k,i}v_i = \bar{r}_k - r_f \quad k = 1, 2, \ldots, n
$$
\[
\begin{align*}
2v_1 + v_2 &= .2 \\
v_1 + 2v_2 + v_3 &= .6 \\
v_2 + 2v_3 &= .6
\end{align*}
\]

Thus

\[v_1 = .2, \; v_2 = -.2, \; v_3 = -.4.\]

Then we put

\[w_i = \frac{v_i}{\sum_{k=1}^{n} v_k}\]

yielding

\[w_1 = .5, \; w_2 = -.5, \; w_3 = 1.\]

Q6 Solve the following quadratic programming problem:

Minimise

\[(x_1 + x_2)^2 + (x_1 - x_3)^2 + x_3^2.\]

Subject to

\[x_1 + x_2 + x_3 = 1 \text{ and } x_1, x_2, x_3 \geq 0.\]

Solution
\[ y = \frac{1}{2} \cos^2 x + \frac{1}{2} \sin^2 x = \frac{1}{2} \quad \text{subject to} \quad \sin x + \cos x = 1 \]

\[ \text{numerators (c} + \cos x)^2 + (\sin x - c)^2 + \cos x \]
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$$\frac{y}{1} = \epsilon \times \epsilon \times 0 = 0$$

Optimal solution

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