Polya’s Theory of Counting
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Example 1 A disc lies in a plane. Its centre is fixed but it is free to rotate. It has been divided into \( n \) sectors of angle \( \frac{2\pi}{n} \). Each sector is to be colored Red or Blue. How many different colorings are there?

One could argue for \( 2^n \).

On the other hand, what if we only distinguish colorings which cannot be obtained from one another by a rotation. For example if \( n = 4 \) and the sectors are numbered 0,1,2,3 in clockwise order around the disc, then there are only 6 ways of coloring the disc – 4R, 4B, 3R1B, 1R3B, RRBB and RBRB.
Now consider an $n \times n$ “chessboard” where $n \geq 2$. Here we color the squares Red and Blue and two colorings are different only if one cannot be obtained from another by a rotation or a reflection. For $n = 2$ there are 6 colorings.
The general scenario that we consider is as follows: We have a set $X$ which will stand for the set of colorings when transformations are not allowed. (In example 1, $|X| = 2^n$ and in example 2, $|X| = 2^{n^2}$).

In addition there is a set $G$ of permutations of $X$. This set will have a group structure:

Given two members $g_1, g_2 \in G$ we can define their composition $g_1 \circ g_2$ by $g_1 \circ g_2(x) = g_1(g_2(x))$ for $x \in X$. We require that $G$ is closed under composition i.e. $g_1 \circ g_2 \in G$ if $g_1, g_2 \in G$. 

Generating Functions
We also have the following:

**A1** The *identity* permutation $1_X \in G$.

**A2** $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ (Composition is associative).

**A3** The inverse permutation $g^{-1} \in G$ for every $g \in G$.

(A set $G$ with a binary relation $\circ$ which satisfies **A1,A2,A3** is called a **Group**).
In example 1 \( D = \{0, 1, 2, \ldots, n - 1\} \), \( X = 2^D \) and the group is \( G_1 = \{e_0, e_1, \ldots, e_{n-1}\} \) where \( e_j \ast x = x + j \mod n \) stands for rotation by \( 2j\pi/n \).

In example 2, \( X = 2^{[n]^2} \). We number the squares 1, 2, 3, 4 in clockwise order starting at the upper left and represent \( X \) as a sequence from \( \{r, b\}^4 \) where for example rrbr means color 1, 2, 4 Red and 3 Blue. \( G_2 = \{e, a, b, c, p, q, r, s\} \) is in a sense independent of \( n \). \( e, a, b, c \) represent a rotation through 0, 90, 180, 270 degrees respectively. \( p, q \) represent reflections in the vertical and horizontal and \( r, s \) represent reflections in the diagonals 1, 3 and 2, 4 respectively.
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From now on we will write $g \ast x$ in place of $g(x)$.

**Orbits:** If $x \in X$ then its orbit

$O_x = \{ y \in X : \exists g \in G \text{ such that } g \ast x = y \}$.

**Lemma 1** The orbits partition $X$.

**Proof** $x = 1_x \ast x$ and so $x \in O_x$ and so $X = \bigcup_{x \in X} O_x$.

Suppose now that $O_x \cap O_y \neq \emptyset$ i.e. $\exists g_1, g_2$ such that $g_1 \ast x = g_2 \ast y$. But then for any $g \in G$ we have

$$g \ast x = (g \circ (g_1^{-1} \circ g_2)) \ast y \in O_y$$

and so $O_x \subseteq O_y$. Similarly $O_y \subseteq O_x$. Thus $O_x = O_y$ whenever $O_x \cap O_y \neq \emptyset$. \qed

Generating Functions
The two problems we started with are of the following form: Given a set $X$ and a group of permutations acting on $X$, compute the number of orbits i.e. distinct colorings.

A subset $H$ of $G$ is called a sub-group of $G$ if it satisfies axioms A1, A2, A3 (with $G$ replaced by $H$).

The stabilizer $S_x$ of the element $x$ is $\{g : g \ast x = x\}$. It is a sub-group of $G$.

- A1: $1_x \ast x = x$.
- A3: $g, h \in S_x$ implies $(g \circ h) \ast x = g \ast (h \ast x) = g \ast x = x$.

A2 holds for any subset.
Lemma 2
If $x \in X$ then $|O_x| |S_x| = |G|$.

Proof Fix $x \in X$ and define an equivalence relation $\sim$ on $G$ by

$$g_1 \sim g_2 \text{ if } g_1 \ast x = g_2 \ast x.$$ 

Let the equivalence classes be $A_1, A_2, \ldots, A_m$. We first argue that

$$|A_i| = |S_x| \quad i = 1, 2, \ldots, m. \quad (1)$$

Fix $i$ and $g \in A_i$. Then

$$h \in A_i \iff g \ast x = h \ast x \iff (g^{-1} \circ h) \ast x = x$$

$$\iff (g^{-1} \circ h) \in S_x \iff h \in g \circ S_x$$

where $g \circ S_x = \{g \circ \sigma : \sigma \in S_x\}$. 

Generating Functions
Thus \(|A_i| = |g \circ S_x|\). But \(|g \circ S_x| = |S_x|\) since if \(\sigma_1, \sigma_2 \in S_x\) and 
\(g \circ \sigma_1 = g \circ \sigma_2\) then

\[g^{-1} \circ (g \circ \sigma_1) = (g^{-1} \circ g) \circ \sigma_1 = \sigma_1 = g^{-1} \circ (g \circ \sigma_2) = \sigma_2.\]

This proves (1).

Finally, \(m = |O_x|\) since there is a distinct equivalence class for each distinct \(g \ast x\). \(\square\)
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Let $\nu_{X,G}$ denote the number of orbits.

Theorem 1

$$\nu_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x|.$$ 

Proof

$$\nu_{X,G} = \sum_{x \in X} \frac{1}{|O_x|} = \sum_{x \in X} \frac{|S_x|}{|G|},$$

from Lemma 1. □
Thus in example 1 we have

$$\nu_{X,G} = \frac{1}{4}(4+1+1+1+1+1+1+1+1+2+2+1+1+1+1+4) = 6.$$ 

In example 2 we have

$$\nu_{X,G} = \frac{1}{8}(8+2+2+2+2+2+2+2+4+4+2+2+2+2+8) = 6.$$ 

Theorem 1 is hard to use if $|X|$ is large, even if $|G|$ is small.

For $g \in G$ let $Fix(g) = \{x \in X : g \ast x = x\}$. 

Generating Functions
Theorem 2
(Frobenius, Burnside)

\[ \nu_{X,G} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|. \]

Proof

Let \( A(x, g) = 1_{g \cdot x = x} \). Then

\[ \nu_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x| \]
\[ = \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} A(x, g) \]
\[ = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X} A(x, g) \]
\[ = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|. \]

□
Let us consider example 1 with $n = 6$. We compute

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<td>\text{Fix}(g)</td>
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<td>8</td>
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Applying Theorem 2 we obtain

$$\nu_{X,G} = \frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = 14.$$
Cycles of a permutation

Let $\pi : D \to D$ be a permutation of the finite set $D$. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : i \in D\}$. $\Gamma_\pi$ is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.


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The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$. 

Generating Functions
In general consider the sequence $i, \pi(i), \pi^2(i), \ldots$.

Since $D$ is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that $\pi$ is a permutation.

So $i$ lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$.

If $j$ is not a vertex of $C$ then $\pi(j)$ is not on $C$ and so we can repeat the argument to show that the rest of $D$ is partitioned into cycles.
First consider $e_0, e_1, \ldots, e_{n-1}$ as permutations of $D$.

The cycles of $e_0$ are $(1), (2), \ldots, (n)$.

Now suppose that $0 < m < n$. Let $a_m = \gcd(m, n)$ and $k_m = n/a_m$. The cycle $C_i$ of $e_m$ containing the element $i$ is

$(i, i + m, i + 2m, \ldots, i + (k_m - 1)m)$ since $n$ is a divisor $k_m m$ and not a divisor of $k' m$ for $k' < k_m$. In total, the cycles of $e_m$ are $C_0, C_1, \ldots, C_{a_m - 1}$.

This is because they are disjoint and together contain $n$ elements. (If $i + rm = i' + r'm \mod n$ then

$(r - r')m + (i - i') = \ell n$. But $|i - i'| < a_m$ and so dividing by $a_m$ we see that we must have $i = i'$.)
Next observe that if coloring $x$ is fixed by $e_m$ then elements on the same cycle $C_i$ must be colored the same. Suppose for example that the color of $i + bm$ is different from the color of $i + (b + 1)m$, say Red versus Blue. Then in $e_m(x)$ the color of $i + (b + 1)m$ will be Red and so $e_m(x) \neq x$. Conversely, if elements on the same cycle of $e_m$ have the same color then in $x \in Fix(e_m)$. This property is not peculiar to this example, as we will see.

Thus in this example we see that $|Fix(e_m)| = 2^{am}$ and then applying Theorem 2 we see that

$$\nu_{X,G} = \frac{1}{n} \sum_{m=0}^{n-1} 2^{\gcd(m,n)}.$$
Example 2

It is straightforward to check that when $n$ is even, we have

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For example, if we divide the chessboard into 4 $n/2 \times n/2$ sub-squares, numbered 1,2,3,4 then a coloring is in $\text{Fix}(a)$ iff each of these 4 sub-squares have colorings which are rotations of the coloring in square 1.
Polya’s Theorem

We now extend the above analysis to answer questions like: How many distinct ways are there to color an $8 \times 8$ chessboard with 32 white squares and 32 black squares? The scenario now consists of a set $D$ (Domain, a set $C$ (colors) and $X = \{x : D \rightarrow C\}$ is the set of colorings of $D$ with the color set $C$. $G$ is now a group of permutations of $D$. 

Generating Functions
We see first how to extend each permutation of \( D \) to a permutation of \( X \). Suppose that \( x \in X \) and \( g \in G \) then we define \( g \ast x \) by

\[
g \ast x(d) = x(g^{-1}(d)) \quad \text{for all } d \in D.
\]

**Explanation:** The color of \( d \) is the color of the element \( g^{-1}(d) \) which is mapped to it by \( g \).

Consider Example 1 with \( n = 4 \). Suppose that \( g = e_1 \) i.e. rotate clockwise by \( \pi/2 \) and \( x(1) = b, x(2) = b, x(3) = r, x(4) = r \). Then for example

\[
g \ast x(1) = x(g^{-1}(1)) = x(4) = r, \text{ as before.}
\]
Now associate a weight \( w_c \) with each \( c \in C \).

If \( x \in X \) then

\[
W(x) = \prod_{d \in D} w_{x(d)}.
\]

Thus, if in Example 1 we let \( w(r) = R \) and \( w(b) = B \) and take

\( x(1) = b, x(2) = b, x(3) = r, x(4) = r \)

then we will write

\[
W(x) = B^2 R^2.
\]

For \( S \subseteq X \) we define the inventory of \( S \) to be

\[
W(S) = \sum_{x \in S} W(x).
\]

The problem we discuss now is to compute the pattern inventory \( PI = W(S^*) \) where \( S^* \) contains one member of each orbit of \( X \) under \( G \).
For example, in the case of Example 2, with \( n = 2 \), we get
\[
\]

To see that the definition of \( PL \) makes sense we need to prove

**Lemma 3** If \( x, y \) are in the same orbit of \( X \) then \( W(x) = W(y) \).

**Proof** Suppose that \( g \ast x = y \). Then
\[
W(y) = \prod_{d \in D} w_y(d)
= \prod_{d \in D} w_{g \ast x}(d)
= \prod_{d \in D} w_x(g^{-1}(d)) \quad (2)
= \prod_{d \in D} w_x(d) \quad (3)
= W(x)
\]

Note, that we can go from (2) to (3) because as \( d \) runs over \( D \), \( g^{-1}(d) \) also runs over \( d \).
Let $\Delta = |D|$. If $g \in G$ has $k_i$ cycles of length $i$ then we define

$$ct(g) = x_1^{k_1}x_2^{k_2} \cdots x_\Delta^{k_\Delta}.$$ 

The **Cycle Index Polynomial** of $G$, $C_G$ is then defined to be

$$C_G(x_1, x_2, \ldots, x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g).$$

In Example 2 with $n = 2$ we have

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ct(g)$</td>
<td>$x_1^4$</td>
<td>$x_4$</td>
<td>$x_2^2$</td>
<td>$x_4$</td>
<td>$x_2^2$</td>
<td>$x_2^2$</td>
<td>$x_1^2x_2$</td>
<td>$x_1^2x_2$</td>
</tr>
</tbody>
</table>

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2x_2 + 2x_4).$$
In Example 2 with $n = 3$ we have

<table>
<thead>
<tr>
<th>$g$</th>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ct(g)$</td>
<td>$x_1^9$</td>
<td>$x_1 x_4^2$</td>
<td>$x_1 x_2^4$</td>
<td>$x_1 x_4^2$</td>
<td>$x_1^3 x_2^3$</td>
<td>$x_1^3 x_2^3$</td>
<td>$x_1^3 x_2^3$</td>
<td>$x_1^3 x_2^3$</td>
</tr>
</tbody>
</table>

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^9 + x_1 x_2^4 + 4x_1^3 x_2^3 + 2x_1 x_4^2).$$
Theorem (Polya)

\[ PI = C_G \left( \sum_{c \in C} w_c, \sum_{c \in C} w_c^2, \ldots, \sum_{c \in C} w_c^\Delta \right) . \]

Proof In Example 2, we replace \( x_1 \) by \( R + B \), \( x_2 \) by \( R^2 + B^2 \) and so on. When \( n = 2 \) this gives

\[
PI = \frac{1}{8} \left( (R + B)^4 + 3(R^2 + B^2)^2 + \right.
\]

\[
\left. 2(R + B)^2(R^2 + B^2) + 2(R^4 + B^4) \right)
\]

\[ = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4. \]

Putting \( R = B = 1 \) gives the number of distinct colorings. Note also the formula for \( PI \) tells us that there are 2 distinct colorings using 2 reds and 2 Blues.
Proof of Polya’s Theorem
Let $X = X_1 \cup X_2 \cup \cdots \cup X_m$ be the equivalence classes of $X$ under the relation

$$x \sim y \text{ iff } W(x) = W(y).$$

By Lemma 2, $g \ast x \sim x$ for all $x \in X, g \in G$ and so we can think of $G$ acting on each $X_i$ individually i.e. we use the fact that $x \in X_i$ implies $g \ast x \in X_i$ for all $i \in [m], g \in G$. We use the notation $g^{(i)} \in G^{(i)}$ when we restrict attention to $X_i$. 

Generating Functions
Let $m_i$ denote the number of orbits $\nu_{X_i,G^{(i)}}$ and $W_i$ denote the common PI of $G^{(i)}$ acting on $X_i$. Then

\[
PI = \sum_{i=1}^{m} m_i W_i
\]

\[
= \sum_{i=1}^{m} W_i \left( \frac{1}{|G|} \sum_{g \in G} |Fix(g^{(i)})| \right) \quad \text{by Theorem 2}
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} |Fix(g^{(i)})| W_i
\]

\[
= \frac{1}{|G|} \sum_{g \in G} W(Fix(g))
\]

(4)

Note that (4) follows from $Fix(g) = \bigcup_{i=1}^{m} Fix(g^{(i)})$ since $x \in Fix(g^{(i)})$ iff $x \in X_i$ and $g \ast x = x$. 

Generating Functions
Suppose now that $ct(g) = x_1^{k_1}x_2^{k_2} \cdots x_\Delta^{k_\Delta}$ as above. Then we claim that

$$W(Fix(g)) = \left(\sum_{c \in C} w_c\right)^{k_1} \left(\sum_{c \in C} w_c^2\right)^{k_2} \cdots \left(\sum_{c \in C} w_c^\Delta\right)^{k_\Delta}. \quad (5)$$

Substituting (5) into (4) yields the theorem.

To verify (5) we use the fact that if $x \in Fix(g)$, then the elements of a cycle of $g$ must be given the same color. A cycle of length $i$ will then contribute a factor $\sum_{c \in C} w_c^i$ where the term $w_c^i$ comes from the choice of color $c$ for every element of the cycle.  \[\square\]