SOME EXTREMAL PROBLEMS
Let $\mathcal{P}_n = \{A : A \subseteq [n]\}$ denote the power set of $[n]$.

$\mathcal{A} \subseteq \mathcal{P}_n$ is a Sperner family if $A, B \in \mathcal{A}$ implies that $A \nsubseteq B$ and $B \nsubseteq A$.

**Theorem**

*If $\mathcal{A} \subseteq \mathcal{P}_n$ is a Sperner family $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.***

**Proof**

We will show that

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1. \quad (1)$$

Now $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$ for all $k$ and so

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{A}|}{\binom{n}{\lfloor n/2 \rfloor}}.$$
Proof of (1): Let $\pi$ be a random permutation of $[n]$.

For a set $A \in \mathcal{A}$ let $\mathcal{E}_A$ be the event

$$\{\pi(1), \pi(2), \ldots, \pi(|A|)\} = A.$$ 

If $A, B \in \mathcal{A}$ then the events $\mathcal{E}_A, \mathcal{E}_B$ are disjoint.

So

$$\sum_{A \in \mathcal{A}} \Pr(\mathcal{E}_A) \leq 1.$$ 

On the other hand, if $A \in \mathcal{A}$ then

$$\Pr(\mathcal{E}_A) = \frac{|A|!(n-|A|)!}{n!} = \frac{1}{\binom{n}{|A|}}$$

and (1) follows.
The set of all sets of size $\lceil n/2 \rceil$ is a Sperner family and so the bound in the above theorem is best possible.

Inequality (1) can be generalised as follows: Let $s \geq 1$ be fixed. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that there do not exist distinct $A_1, A_2, \ldots, A_{s+1} \in \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$.

**Theorem**

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$  

**Proof**  Let $\pi$ be a random permutation of $[n]$.

Let $\mathcal{E}(A)$ be the event $\{\{\pi(1), \pi(2), \ldots, \pi(|A|) = A\}\}$. 

Some extremal problems
Let 
\[ Z_i = \begin{cases} 
1 & \text{if } \mathcal{E}(A_i) \text{ occurs.} \\
0 & \text{otherwise.} 
\end{cases} \]

and let \( Z = \sum_i Z_i \) be the number of events \( \mathcal{E}(A_i) \) that occur.

Now our family is such that \( Z \leq s \) for all \( \pi \) and so
\[
E(Z) = \sum_i E(Z_i) = \sum_i \Pr(\mathcal{E}(A_i)) \leq s.
\]

On the other hand, \( A \in \mathcal{A} \) implies that \( \Pr(\mathcal{E}(A)) = \frac{1}{\binom{n}{|A|}} \) and the required inequality follows. \( \square \)
A family $A \subseteq P_n$ is an *intersecting* family if $A, B \in A$ implies $A \cap B \neq \emptyset$.

**Theorem**

*If $A$ is an intersecting family then $|A| \leq 2^{n-1}$.***

**Proof**  
Pair up each $A \in P_n$ with its complement $A^c = [n] \setminus A$. This gives us $2^{n-1}$ pairs altogether. Since $A$ is intersecting it can contain at most one member of each pair.

If $A = \{ A \subseteq [n] : 1 \in A \}$ then $A$ is intersecting and $|A| = 2^{n-1}$ and so the above theorem is best possible.
Theorem

If $A$ is an intersecting family and $A \in A$ implies that $|A| = k \leq \lfloor n/2 \rfloor$ then

$$|A| \leq \binom{n-1}{k-1}$$

Proof

If $\pi$ is a permutation of $[n]$ and $A \subseteq [n]$ let

$$\theta(\pi, A) = \begin{cases} 1 & \exists s : \{\pi(s), \pi(s+1), \ldots, \pi(s+k-1)\} = A \\ 0 & \text{otherwise} \end{cases}$$

where $\pi(i) = \pi(i-n)$ if $i > n$.

We will show that for any permutation $\pi$,

$$\sum_{A \in A} \theta(\pi, A) \leq k. \quad (2)$$

Some extremal problems
Assume (2). We first observe that if $\pi$ is a random permutation then
\[
\mathbb{E}(\theta(\pi, A)) = n \frac{k!(n-k)!}{n!} = \frac{k}{\binom{n-1}{k-1}}
\]
and so, from (2),
\[
k \geq \mathbb{E}\left( \sum_{A \in \mathcal{A}} \theta(\pi, A) \right) = \sum_{A \in \mathcal{A}} \frac{k}{\binom{|A|-1}{n-1}}
\]
Hence
\[
|\mathcal{A}| \leq \binom{n-1}{k-1}
\]
Some extremal problems
Assume w.l.o.g. that $\pi$ is the identity permutation.

Let $A_t = \{t, t + 1, \ldots, t + k - 1\}$ and suppose that $A_s \in A$.

All of the other sets $A_t$ that intersect $A_s$ can be partitioned into pairs $A_{s-i}, A_{s+k-i}$, $1 \leq i \leq k - 1$ and the members of each pair are disjoint. Thus $A$ can contain at most one from each pair. This verifies (2).
Kraft’s Inequality

Let $x_1, x_2, \ldots, x_m$ be a collection of sequences over an alphabet $\Sigma$ of size $s$. Let $x_i$ have length $n_i$ and let $n = \max\{n_1, n_2, \ldots, n_m\}$.

Assume next that no sequence is a prefix of any other sequence: Sequence $x_i = a_1 a_2 \cdots a_{n_i}$ is a prefix of $x_j = b_1 b_2 \cdots b_{n_j}$ if $a_i = b_i$ for $i = 1, 2, \ldots, n_i$.

**Theorem**

$$\sum_{i=1}^{m} r^{-n_i} \leq 1.$$
**Proof:** Let $x$ be a random sequence of length $n$. Let $E_i$ be the event $x_i$ is a prefix of $x$. Then

(a) $\Pr(E_i) = r^{-n_i}$.
(b) The event $E_i, i = 1, 2, \ldots, m$ are disjoint.
   (If $E_i$ and $E_j$ both occur and $n_i \leq n_j$ then $x_i$ is a prefix of $x_j$.

Property (b) implies that

$$\Pr\left(\bigcup_{i=1}^{m} E_i\right) = \Pr(E_1) + \Pr(E_2) + \cdots + \Pr(E_m) \leq 1.$$ 

The theorem now follows from Property (a).
The trace of a set system

Let $X$ be a set and suppose that $\mathcal{F} \subseteq 2^X$.

For $Y \subseteq X$ we let $\mathcal{F} \cap Y = \{ F \cap Y : F \in \mathcal{F} \}$.

Then for positive integer $k$ we let

$$f_{\mathcal{F}}(k) = \max \left\{ |\mathcal{F} \cap Y| : Y \in \binom{X}{k} \right\}.$$

We define the trace number of the system $\mathcal{F}$ by

$$tr(\mathcal{F}) = \max \{ m : f_{\mathcal{F}}(m) = 2^m \}.$$
Theorem

Suppose that $|X| = n$ and $\mathcal{F} \subseteq 2^X$ and $|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}$. Then $tr(\mathcal{F}) \geq k$.

Proof

For $x \in X$ set

$$\mathcal{F}_x = \mathcal{F} \cap (X \setminus \{x\}) = \{A \setminus \{x\} : A \in \mathcal{F}\}.$$ 

Let $\phi_x : \mathcal{F} \to \mathcal{F}_x$ be given by $\phi_x(A) = A \setminus \{x\}$.

$\phi_x$ is onto and if $|\phi^{-1}(B)| \geq 2$ then $\phi^{-1}(B) = \{B, B \cup \{x\}\}$ and $x \notin B$. 

Some extremal problems
Let

\[ A_x = \{ A \in \mathcal{F} : x \in A, A \setminus \{x\} \in \mathcal{F}\} . \]

\[ B_x = \{ B \in \mathcal{F} : x \notin B, B \cup \{x\} \in \mathcal{F}\} . \]

Then

\[ |\mathcal{F}| - |\mathcal{F}_x| = |A_x| = |B_x|. \]  

(3)

Note that if \( tr(B_x) \geq k - 1 \) then \( tr(\mathcal{F}) \geq k \). Indeed, suppose \( B_x \cap Y = 2^Y \) where \( |Y| = k - 1 \). Set \( Z = Y \cup \{x\} \). Then

\[ F \cap Z \supset (A_x \cup B_x) \cap Z = 2^Z . \]

Because if \( x \in U \subset Z \) then \( U \setminus \{x\} = B \cap Y = B \cap Z \) for some \( B \in B_x \) by assumption. So \( U = A \cap Z \) where \( A = B \cup \{x\} \in A_x \).
To complete the proof we use induction on $n + k$. For $n + k = 1$ there is nothing to prove.

Suppose that $n + k \geq 2$ and the result is true for smaller values of $n + k$.

Let $x \in X$. If $|F_x| > \sum_{j=0}^{k-1} \binom{n-1}{j}$ then $tr(F_x) \geq k$ by induction and so $tr(F_x) \geq k$. Otherwise, by (3),

\[
|B_x| = |F| - |F_x| > \sum_{j=0}^{k-1} \binom{n}{j} - \sum_{j=0}^{k-1} \binom{n-1}{j} = \sum_{j=1}^{k-1} \binom{n-1}{j-1} = \sum_{j=0}^{k-2} \binom{n-1}{j}.
\]

Hence, by induction, $tr(B_x) \geq k - 1$ and so $tr(B) \geq k$. □
Corollary

If $\mathcal{F}$ is a family of subsets of an infinite set $S$ then either $f_{\mathcal{F}}(k) = 2^k$ for every $k$ or else there exists $\ell$ such that $f_{\mathcal{F}}(n) \leq n^\ell$ for every $n \geq \ell$.

Proof  Suppose that $f_{\mathcal{F}}(k) \neq 2^\ell$ for some $\ell$. Then by the theorem,

$$f_{\mathcal{F}}(n) \leq \sum_{j=0}^{\ell-1} \binom{n}{j} \leq n^\ell \text{ for } n > \ell.$$
Sunflowers

A sunflower of size $r$ is a family of sets $A_1, A_2, \ldots, A_r$ such that every element that belongs to more than one of the sets belongs to all of them.

Let $f(k, r)$ be the maximum size of a family of $k$-sets without a sunflower of size $r$.

**Theorem**

$f(k, r) \leq (r - 1)^k k!$.

**Proof**

Let $\mathcal{F}$ be a family of $k$-sets without a sunflower of size $r$. Let $A_1, A_2, \ldots, A_t$ be a maximum subfamily of pairwise disjoint subsets in $\mathcal{F}$.

Since a family of pairwise disjoint is a sunflower, we must have $t < r$. 

Some extremal problems
Now let $A = \bigcup_{i=1}^{t} A_i$. For every $a \in A$ consider the family

$$\mathcal{F}_a = \{S \setminus \{a\} : S \in \mathcal{F}, a \in S\}.$$ 

Now the size of $A$ is at most $(r - 1)k$.

The size of each $\mathcal{F}_a$ is at most $f(k - 1, r)$. This is because a sunflower in $\mathcal{F}_a$ is a sunflower in $\mathcal{F}$.

So,

$$f(k, r) \leq (r - 1)k \times f(k - 1, r) \leq (r - 1)k \times (r - 1)^{k-1}(k - 1)!,$$

by induction. $\square$
Matchings

A matching $M$ of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.

Some extremal problems
An $M$-alternating path joining 2 $M$-unsaturated vertices is called an $M$-augmenting path.

(a,b,c,d,e,f) is an $M$-alternating path

Some extremal problems
**Theorem**

A matching $M$ is a maximum matching of $G$ if no matching $M'$ has more edges.

**Proof**

Suppose $M$ has an augmenting path $P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where $e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k + 1$ and $f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.

$$M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}.$$
\[ |M'| = |M| + 1. \]

\( M' \) is a matching

For \( x \in V \) let \( d_M(x) \) denote the degree of \( x \) in matching \( M \), So

\[
\begin{align*}
    d_M(x) & \quad x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\
    d_M(x) & \quad x \in \{b_1, \ldots, a_k\} \\
    d_M(x) + 1 & \quad x \in \{a_0, b_{k+1}\}
\end{align*}
\]

So if \( M \) has an augmenting path it is not maximum.
Suppose $M$ is not a maximum matching and $|M'| > |M|$. Consider $H = G[M \uplus M']$ where $M \uplus M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in exactly one of $M, M'$. Maximum degree of $H$ is $2 - \leq 1$ edge from $M$ or $M'$. So $H$ is a collection of vertex disjoint alternating paths and cycles.

(a) (b) (c) (d) x,y M-unsaturated

$|M'| > |M|$ implies that there is at least one path of type (d). Such a path is $M$-augmenting □

Some extremal problems
Bipartite Graphs

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition $A, B$. For $S \subseteq A$ let $N(S) = \{ b \in B : \exists a \in S, (a, b) \in E \}$.

Clearly, $|M| \leq |A|, |B|$ for any matching $M$ of $G$. 

Some extremal problems
Systems of Distinct Representatives

Let $S_1, S_2, \ldots, S_m$ be arbitrary sets. A set $s_1, s_2, \ldots, s_m$ of $m$ distinct elements is a system of distinct representatives if $s_i \in S_i$ for $i = 1, 2, \ldots, m$.

For example $\{1, 2, 4\}$ is a system of distinct representatives for $\{1, 2, 3\}, \{2, 5, 6\}, \{2, 4, 5\}$.

Now define the bipartite graph $G$ with vertex bipartition $[m], S$ where $S = \bigcup_{i=1}^{m} S_i$ and an edge $(i, s)$ iff $s \in S_i$.

Then $S_1, S_2, \ldots, S_m$ has a system of distinct representatives iff $G$ has a matching of size $m$. 
Hall’s Theorem

**Theorem**

*G contains a matching of size** \(|A|\) iff

\[ |N(S)| \geq |S| \quad \forall S \subseteq A. \] (4)

\[ N(\{a_1, a_2, a_3\}) = \{b_1, b_2\} \] and so at most 2 of \(a_1, a_2, a_3\) can be saturated by a matching.
Only if: Suppose $M = \{(a, \phi(a)) : a \in A\}$ saturates $A$.

If: Let $M = \{(a, \phi(a)) : a \in A'\}$ ($A' \subseteq A$) is a maximum matching. Suppose $a_0 \in A$ is $M$-unsaturated. We show that (4) fails.

\[ |N(S)| \geq \left| \{\phi(s) : s \in S\} \right| = |S| \]
Let
\[ A_1 = \{ a \in A : \text{such that } a \text{ is reachable from } a_0 \text{ by an } M\text{-alternating path.} \} \]
\[ B_1 = \{ b \in B : \text{such that } b \text{ is reachable from } a_0 \text{ by an } M\text{-alternating path.} \} \]

Some extremal problems
• $B_1$ is $M$-saturated else there exists an $M$-augmenting path.
• If $a \in A_1 \setminus \{a_0\}$ then $\phi(a) \in B_1$.

\[
\begin{array}{c}
\text{a}_0 \\
\phi(a) \\
a
\end{array}
\]

• If $b \in B_1$ then $\phi^{-1}(b) \in A_1 \setminus \{a_0\}$.
So $|B_1| = |A_1| - 1$.

\[
\begin{array}{c}
\text{a}_0 \\
\text{b} \\
a
\end{array}
\]

So $|N(A_1)| = |A_1| - 1$ and (4) fails to hold.
Marriage Theorem

**Theorem**

Suppose $G = (A \cup B, E)$ is $k$-regular. ($k \geq 1$) i.e. $d_G(v) = k$ for all $v \in A \cup B$. Then $G$ has a perfect matching.

**Proof**

$k|A| = |E| = k|B|$ and so $|A| = |B|$. Suppose $S \subseteq A$. Let $m$ be the number of edges incident with $S$. Then $k|S| = m \leq k|N(S)|$. So (4) holds and there is a matching of size $|A|$ i.e. a perfect matching.
Edge Covers

A set of vertices $X \subseteq V$ is a covering of $G = (V, E)$ if every edge of $E$ contains at least one endpoint in $X$.

Lemma

If $X$ is a covering and $M$ is a matching then $|X| \geq |M|$.

Proof

Let $M = \{(a_1, b_i) : 1 \leq i \leq k\}$. Then $|X| \geq |M|$ since $a_i \in X$ or $b_i \in X$ for $1 \leq i \leq k$ and $a_1, \ldots, b_k$ are distinct. □
Konig’s Theorem

Let $\mu(G)$ be the maximum size of a matching. Let $\beta(G)$ be the minimum size of a covering. Then $\mu(G) \leq \beta(G)$.

**Theorem**

*If G is bipartite then $\mu(G) = \beta(G)$.***

**Proof**  
Let $M$ be a maximum matching. Let $S_0$ be the $M$-unsaturated vertices of $A$. Let $S \supseteq S_0$ be the $A$-vertices which are reachable from $S_0$ by $M$-alternating paths. Let $T$ be the $M$-neighbours of $S \setminus S_0$. Some extremal problems
Let $X = (A \setminus S) \cup T$.

- $|X| = |M|$.
- $|T| = |S \setminus S_0|$. The remaining edges of $M$ cover $A \setminus S$ exactly once.
- $X$ is a cover.

There are no edges $(x, y)$ where $x \in S$ and $y \in B \setminus T$. Otherwise, since $y$ is $M$-saturated (no $M$-augmenting paths) the $M$-neighbour of $y$ would have to be in $S$, contradicting $y \notin T$. \[\square\]