1. How many ways are there of \( k \)-coloring the squares of the above diagram if the group acting is \( e_0, e_1, e_2, e_3 \) where \( e_j \) is rotation by \( 2\pi j/4 \). Assume that instead of 28 squares there are \( 4n - 4 \).

Solution:

\[
\begin{array}{cccc}
g & e_0 & e_1 & e_2 & e_3 \\
|\text{Fix}(g)| & k^{4n-4} & k^{n-1} & k^{2n-2} & k^{n-1}
\end{array}
\]

So the total number of colorings is

\[
\frac{k^{4n-4} + k^{n-1} + k^{2n-2} + k^{n-1}}{4}.
\]

2. How many ways are there of \( k \)-coloring the squares of the same diagram if the group acting is \( e_0, e_1, e_2, e_3, p, q, r, s \) where \( p, q, r, s \) are horizontal, vertical, diagonal reflections.

Solution:

\( n \) even:

\[
\begin{array}{cccccccc}
g & e_0 & e_1 & e_2 & e_3 & p & q & r & s \\
|\text{Fix}(g)| & k^{4n-4} & k^{n-1} & k^{2n-2} & k^{n-1} & k^{2n-2} & k^{2n-2} & k^{2n-2} & k^{2n-1} & k^{2n-1}
\end{array}
\]

So the total number of colorings is

\[
\frac{k^{4n-4} + k^{n-1} + k^{2n-2} + k^{n-1} + k^{2n-2} + k^{2n-2} + k^{2n-1} + k^{2n-1}}{8}.
\]

\( n \) odd

\[
\begin{array}{cccccccc}
g & e_0 & e_1 & e_2 & e_3 & p & q & r & s \\
|\text{Fix}(g)| & k^{4n-4} & k^{n-1} & k^{2n-2} & k^{n-1} & k^{2n-1} & k^{2n-1} & k^{2n-1} & k^{2n-1}
\end{array}
\]

So the total number of colorings is

\[
\frac{k^{4n-4} + k^{n-1} + k^{2n-2} + k^{n-1} + k^{2n-1} + k^{2n-1} + k^{2n-1} + k^{2n-1}}{8}.
\]
3. How many ways are there to arrange 2 M’s, 4 A’s, 5 T’s and 6 H’s under the condition that any arrangement and its reversal are to be considered the same.

Solution: The group $G$ consists of $\{e, a\}$ where $a$ is a reflection through the middle of the word. Now

$$|Fix(e)| = \frac{17!}{2!4!5!6!} = 85765680$$

$$|Fix(a)| = \frac{8!}{1!2!2!3!} = 1680$$

A sequence is in $Fix(a)$ if it is a palindrome i.e. looks the same backwards as forwards. It must have middle letter T. Then we arrange 1 M, 2 A’s, 2 T’s and 3 H’s in any order and then complete the sequence uniquely to a palindrome.

Thus by Burnside’s theorem, the number of sequences is $\frac{85765680 + 1680}{2} = 42883680$. 