1. In a take-away game, the set $S$ of the possible numbers of chips to remove is finite. Show that the Grundy numbers $g$ satisfy $g(n) \leq |S|$ where $n$ is the number of chips remaining.

**Solution:** Observe that for any finite set $A$, $\text{mex}(A) \leq |A|$ since $\text{mex}(A) > |A|$ implies that $A \subseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.

2. Consider the following take-away game: In the first move you are not allowed to take the whole pile. After that, if a player removes $x$ chips, then the next player can remove up to $\lfloor 5x/4 \rfloor$ chips. Determine the $P$ positions.

**Solution:** The $P$-positions, $\{H_1, H_2, \ldots, \}$ satisfy the recurrence
\[ H_{j+1} = H_j + H_k \text{ where } k = \min_{0 \leq \ell \leq j} \{\ell : H_j \leq \lfloor 5H_\ell/4 \rfloor\}. \] (1)

The first 8 values are given by
\[
\begin{array}{cccccccc}
    j & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
    H_j & 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 \\
\end{array}
\]

We can see that $H_j = 2^j$, but we must prove this by induction. But this follows from
\[ \lfloor 5 \times 2^{j-1}/4 \rfloor < 2^j \]
which implies that $k = j$ in (1).

3. Find the set of $P$-positions for the take-away games with subtraction sets
   
   (a) $S = \{1, 3, 7\}$.
   (b) $S = \{1, 4, 6\}$.

Suppose now that there are two piles and the rules for each pile are as above. Now find the $P$ positions for the two pile game.

**Solution:**

(a) The first few numbers are
\[
\begin{array}{cccccccc}
    j & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
    g_1(j) & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

It is apparent that $g_1(j) = j \mod 2$ and this follows by an easy induction: If $j$ is even then $j - x, x \in S$ is odd and if $j$ is odd then $j - x, x \in S$ is even.

(b) The first few numbers are
\[
\begin{array}{cccccccccccc}
    j & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
    g_2(j) & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 2 \\
\end{array}
\]
So, we see that the pattern 0 1 0 1 2 repeats itself. Again, induction can be used to verify that this continues indefinitely.

(c) The $P$-positions are those $j, k$ for which $g_1(j) \oplus g_2(k) = 0$. Thus

$$ P = \{(j, k) : (k \mod 10 \leq 3) \text{ and } (j \mod 2 = k \mod 10)\}. $$