1. Let \( r_n = r(3, 3, \ldots, 3) \) be the minimum integer such that if we \( n \)-color the edges of the complete graph \( K_N \) there is a monochromatic triangle.

(a) Show that \( r_n \leq n(r_{n-1} - 1) + 2 \).

(b) Using \( r_2 = 6 \), show that \( r_n \leq \lceil n!e \rceil + 1 \).

**Solution:** Let \( N = n(r_{n-1} - 1) + 2 \) and consider an \( n \)-coloring \( \sigma \) of the edges of \( K_N \). Now consider the \( N-1 \) edges incident to vertex \( N \). There must be a color, \( n \) say, that is used at least \( r_{n-1} \) times, Pigeon Hole Principle. Now let \( V \subseteq [N-1] \) denote the set of vertices \( v \) for which the edge \( \{v, N\} \) is colored \( n \). Consider the coloring of the edges of \( V \) induced by \( \sigma \). If one of these \( \{v_1, v_2\} \) has color \( N \) then it makes a triangle \( v_1, v_2, N \) with 3 edges colored \( n \). Otherwise the edges of \( V \) only use \( n-1 \) colors and since \( |V| \geq r_{n-1} \) we see by induction that \( V \) contains a mono-chromatic triangle.

(b) Using \( r_2 = 6 \), show that \( r_n \leq \lceil n!e \rceil + 1 \).

**Solution:** Divide the inequality (a) by \( n! \) and putting \( s_n = r_n/n! \) we obtain

\[
s_n \leq s_{n-1} - \frac{1}{(n-1)!} + \frac{2}{n!}.
\]

We write this as

\[
s_n - s_{n-1} \leq -\frac{1}{(n-1)!} + \frac{2}{n!}
\]

\[
s_{n-1} - s_{n-2} \leq -\frac{1}{(n-2)!} + \frac{2}{(n-1)!}
\]

\[
s_3 - s_2 \leq -\frac{1}{1!} + \frac{2}{2!}
\]

Summing gives

\[
s_n - s_2 \leq -1 + \frac{1}{n!} + \sum_{k=2}^{n} \frac{1}{k!} \leq -1 + \frac{1}{n!} + e - 2.
\]

Now \( s_2 = 3 \) and multiplying the above by \( n! \) gives \( r_n \leq n!e + 1 \). We round down, as \( r_n \) is an integer.

2. Show that if the edges of \( K_{m+n} \) are colored red and blue then either (i) there is a red path with \( m \) edges or (ii) a vertex of blue degree at least \( n \).

**Solution:** If there is no vertex of blue degree at least \( n \) then the red graph has minimum degree at least \( m \). Let \( P = x_1, x_2, \ldots, x_k \) be a longest path in the red graph. All of \( x_k \)'s neighbors in the red graph lie on \( P \), else \( P \) can be extended. But \( x_k \) has at least \( m \) neighbours and so \( k \geq m + 1 \).
3. Given a set $I$ of $n$ intervals in $\mathbb{R}$, assume that there is no nested set of intervals with size $k$ (a set of intervals are nested if for every pair, one is completely contained inside the other). Then prove that there exists a subset of size $\lceil n/k \rceil$ where no pair of intervals are nested.

Solution: The nesting property defines a partial order. By Dilworth's theorem, if the longest chain has size $k$, the set of intervals can be partitioned into $k$ sets where each set is an anti-chain. One such anti-chain has size at least $\lceil n/k \rceil$. 