Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2012: Test 2

Name:

Andrew ID:

Write your name and Andrew ID on every page.

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Q1: (40pts)
The sequence \(a_0, a_1, \ldots, a_n, \ldots\) satisfies the following: \(a_0 = 1, a_1 = 9\) and
\[a_n = 6a_{n-1} - 9a_{n-2}\]
for \(n \geq 2\).

(a): Find the generating function \(a(x) = \sum_{n=0}^{\infty} a_n x^n\).

(b): Find an expression for \(a_n, n \geq 0\).

Answer:
\[
\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (6a_{n-1} - 9a_{n-2}) x^n
\]
\[
a(x) - 9x - 1 = 6x(a(x) - 1) - 9x^2a(x)
\]
\[
a(x) = \frac{3x + 1}{9x^2 - 6x + 1} = \frac{3x + 1}{(1 - 3x)^2}
\]
\[
a(x) = \sum_{n=0}^{\infty} \binom{n+1}{1} (3x)^n + 3x \sum_{n=0}^{\infty} \binom{n+1}{1} (3x)^n
\]
\[
a_n = (n + 1)3^n + n3^n = 3^n + 2n3^n
\]
Q2: (40pts)
Recall that a tournament is an oriented complete graph. A Hamiltonian path in a tournament is a directed path that visits every vertex exactly once. Show that there is a tournament of size $n$ with $n!2^{-(n-1)}$ Hamiltonian paths.

**Answer:** Consider a random tournament $T$ and let $X$ be the number of Hamiltonian paths in $T$. For a permutation $\sigma: [n] \to [n]$ define $X_\sigma$ by

$$X_\sigma = \begin{cases} 
1 & \sigma \text{ gives a Hamiltonian path. I.e., } ((\sigma(i), \sigma(i + 1)) \in T, \forall i \in [n]); \\
0 & \text{otherwise.}
\end{cases}$$

Then $X = \sum_\sigma X_\sigma$. Clearly

$$\Pr[X_\sigma] = 2^{-(n-1)}$$

and thus $E_T[X] = n!2^{-(n-1)}$. In particular there is such a tournament.
Q3: (20pts)
A cover in a graph is a set of vertices $C$ such that every vertex not in $C$ has a neighbour in $C$. Show that every graph on $n$ vertices with minimal degree $\delta$ has a cover of size at most $n(1 + \ln(\delta + 1))/(\delta + 1)$.

**Answer:** We follow the same ideas as in finding a cover for the hypercube (the one we used to find a solution for the Hat problem, see slides 20 – 22 in the presentation “The Probabilistic Method”).

We pick a set of vertices at random, where every vertex has probability $p$ of being in the set. Call the set $X$. Let $Y$ be the set of vertices that are not covered by $X$ (i.e. they are not in $X$ and do not have a neighbor in $X$). Clearly $X \cup Y$ is a cover and

$$E[|X \cup Y|] = E[|X| + |Y|] = E[|X|] + E[|Y|] = np + n(1 - p)^{\delta+1}.$$ 

So there is a cover of size at most $n(p + (1 - p)^{\delta+1}) \leq n(p + e^{-p(\delta+1)})$.

The last stage is optimization — finding the probability $p$ that will minimize the size of the cover:

$$(p + e^{-p(\delta+1)})' = 1 - e^{-p(\delta+1)}(\delta + 1) = 0$$

$\Downarrow$

$$p = \frac{\ln(\delta + 1)}{\delta + 1}.$$

Plugging $p$ back in $n(p + e^{-p(\delta+1)})$ we get that the expected cover size is at most

$$n \left( \frac{\ln(\delta + 1) + 1}{\delta + 1} \right).$$

Therefore there is a cover of that size in the graph.