Grundy Numbers

Game represented by $D = (V, A)$. Grundy number $g$ satisfy

$$g(x) = \max \{g(y) : y \in N^+(A)\}$$

Observe

$$x \in P \iff g(x) = 0$$

$$|V| = n$$

$$g(x) = 0 \iff g(y) > 0, \quad \forall y \in N^+(x)$$

$$\forall x \in P \iff y \in V, \quad \forall y \in N^+(x)$$

$$n \in P$$
Grundy Numbers of Simple Game

Rule: 1) A player can remove any even no. of chips but not the whole pile, if it is even.

1) A player can remove whole pile if it is odd.

Terminal positions are 0 & 2.
Claim: $g(0) = 0$
$g(2k) = k - 1$
$g(2k-1) = k$

Proof of claim: by induction on $k$.

True for small $k \leq 4$
Assume true for \( k \leq 4 \).

\[
g(2k) = \min \left\{ g(2k-2), g(2k-4), \ldots, g(2) \right\}
\]

\[
= \min \left\{ k-2, k-3, \ldots, 0 \right\}
\]

\[
= k - 1
\]

\[
g(2k-1) = \min \left\{ g(2k-3), g(2k-5), \ldots, g(1), g(0) \right\}
\]

\[
= \min \left\{ k-1, k-2, \ldots, 1, 0 \right\}
\]

\[
= k.
\]
Sum of Games

\[ x = (x_1, x_2, \ldots, x_p) \] is a position in \( G_1 \oplus G_2 \oplus \cdots \oplus G_p \)

then

\[ g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p) \]

where \( g_i \) is the Grundy function for \( G_i \).

Proof

Can assume that \( p \geq 2 \) and apply induction.

\( p = 3 \)

\[ g(x) = (g_1(x_1) \oplus g_2(x_2)) \oplus g_3(x_3) \]
\[ G = G_1 \oplus G_2 \]

Show \( g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2) \)

[Backwards induction on topological order \( \not\subseteq \mathcal{D}_1 \times \mathcal{D}_2 \) ]

Suppose that \( g_1(x_1) \oplus g_2(x_2) = \mathcal{F} \)

We must show that \( \mathcal{F} = \max \left\{ g_1(x'_1) \oplus g_2(x'_2) : (x'_1, x'_2) \in \mathcal{N}^+(x_1, x_2) \right\} \)

A1: if \( x \in X \) and \( g(x) = \mathcal{F} \Rightarrow a \) then \( \exists x' \in \mathcal{N}^+(x) \) s.t. \( g(x') = a \).

A2: if \( x \in X \) and \( g(x) = \mathcal{F} \) and \( x' \in \mathcal{N}^+(x) \) then \( g(x') \neq \mathcal{F} \).

A3: if \( x \in X \) and \( g(x) = 0 \) and \( x' \in \mathcal{N}^+(x) \) then \( g(x') \neq 0 \).
\[ g_1(x_1) \oplus g_2(x_2) \oplus (a \oplus b) = a \]

\( \text{If } \min \in \mathcal{N}(x_2) \)

\( \text{then } \exists x'_2 \in \mathcal{N}^+(x_2) \text{ s.t.} \)

\[ g_2(x'_2) = g_2(x_2) \oplus d \]

\[ \implies g(x_1, x'_2) = a \]

Show either

\[ g_1(x_1) \oplus d < g_1(x_1) \]

or

\[ g_2(x_2) \oplus d < g_2(x_2) \]
Suppose \( 2^{k-1} \leq d < 2^k \)

\[
\begin{align*}
    d &= \cdots d_k d_{k-1} \cdots d_1 d_0 \\
    &\uparrow \\
    1
\end{align*}
\]

\( d_k = 1 = a_k \oplus b_k \Rightarrow a_k = 0 \quad \& \quad b_k = 1 \)

\[
\left[ a_k = 1 \quad \& \quad b_k = 0 \Rightarrow a > b \right]
\]

\[
1 = b_k = g_1(o_1) \oplus g_2(x_2)
\]

\[
\begin{align*}
    &\text{assume } 1 \\
    &\quad 0
\end{align*}
\]

\[
\rightarrow g_1(o_1) \land d < g_1(x_1) \text{ because } d \text{ "fills" but } k
\]
A1: \[ g_1(x_1') \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2) = b \]

\[ \Rightarrow \]

\[ g_1(x_1') = g_1(x_1) \text{ -- contradiction} \]

\[ g_1 \text{ doesn't "see" same number.} \]
Suppose we have 2 piles a \( S = \{1, 2, 3, 4\} \) in both piles

\[ g_1(n) = n \mod 5 = g_2(n) \quad \text{[Check]} \]

\( (n_1, n_2) \in P \iff n_1 = n_2 \mod 5 \)

\( n_1 \neq n_2 \mod 5 \) then \( g(n_1, n_2) \neq 0 \).


\[ \begin{array}{c|ccc}
\hline
\text{P-position} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \hline
0 & 0 & 0 & 0
\end{array} \]
More complicated Take-away game

$n$ chips

First move, can take $1, 2, \ldots, n-1$ chips.
Subsequently, if player takes $x$ chips,
next player takes $y \leq x$ chips.

Losing positions are powers of 2.
General Strategy:
Take lowest bit.
Can't do this for a power of 2.

# 1's does not decrease