Tic-Tac-Toe - multi-dimensional

The board is $[n] \times [n] \times \ldots \times [n]$ $d$ times

A line is a set of points $\{x^{(1)}, x^{(2)}, \ldots, x^{(d)}\}$

where

$$x^{(1)} = \begin{cases} (k, k, \ldots, k) & k \in [n] \\ (1, 2, \ldots, n) & \text{or} \\ (n, n-1, \ldots, 1) & \end{cases}$$
\[ \begin{array}{ccc}
  & * & * \\
 1,1 & 1,2 & 1,3 \\
  & * & \\
\end{array} \]

\[ \begin{array}{ccc}
  & * & \\
 3,1 & 2,2 & 1,3 \\
  & * & \\
\end{array} \]

\[ x^{(1)} = (1,1,1) \]
\[ x^{(2)} = (1,2,3) \]

\[ \left( n+2 \right)^d \]

\[ \text{Proof} \]

We have to choose \( x^{(1)}, x^{(2)}, \ldots, x^{(d)} \)

\[ n+2 \text{ choices} \]

\[ \left( n+2 \right)^d \]

\[ \frac{\left( n+2 \right)^d}{2} \]

\[ -n^d \]

all constant

2 ways of getting a line
Two players. Each takes a turn in marking a spot on board 0 or X and each aims to get a line.

L2: player 2, can only draw the game

Strategy stealing

If player 2 had a winning strategy, then player 1 could use it. Player 1 plays arbitrarily and then follows player 2 strategy.
Pausing Strategy

$[5]^2$ game

\[
\begin{bmatrix}
11 & 1 & 8 & 1 & 12 \\
6 & 2 & 2 & 9 & 10 \\
3 & 7 & x & 9 & 3 \\
6 & 7 & 4 & 4 & 10 \\
12 & 5 & 8 & 5 & 11
\end{bmatrix}
\]

Every line contains 2 of the same number.

If P1 plays \( i \) then P2 plays the second \( i \).

If P1 plays \( x \) the P2 plays arbitrarily.
**Generalisation**

\[ f = A_1, A_2, \ldots, A_n \leq A \]

A move: a player chooses uncolored member of \( A \) and gives it their own color.

Players win if they are the first player to make \( A_j \) that players color.
Pairing strategy: \( X = \pi x_1 x_2, \ldots, x_{2N-1} x_{2N} \)

such that

\[ A_i = \pi x_{2i-1}, x_{2i} \quad 1 \leq i \leq N \]

Strategy for \( P_2 \): if \( P_1 \) plays \( x_{2i-s} \)

then \( P_2 \) plays \( x_{2i-(1-s)} \)

\[ \bigcup_{i \in S} A_i \geq 2|S| + 4 \quad \forall S \subseteq \{1, \ldots, N\} \]

then there is a pairing strategy
We want a matching of $B$ into $A$.

Then $V_i = a_{2i-1}, a_{2i}$ contained in $A_i$.

Hall's condition reduces to

$$S \subseteq \{b_2, \ldots, N_x^2\} \implies \left| \bigcup_{i \in S} A_i \right| \geq 2 |S|$$

Each $i \in S$ needs to account for $b_{2i-1}, b_{2i}$.
If $|A_i| \geq N$ for $i=1, \ldots, N$ and each $x \in A$ is in at most $N/2$ sets in $F$, then there is a pairing strategy.

**Proof**

$$|S| \times N \leq m \leq |N(S)| \times \frac{N}{2}$$
Tic Tac Toe \( d=2 \)

\( n \) even: each array element \( w_i \leq 3 \) lines

\( n \) odd: each array element \( w_i \leq 4 \) lines

So we immediately see that

\( n \) even & \( n \geq 6 \) or \( n \) odd & \( n \geq 9 \)

then there is a winning strategy.

[ Cases \( n = 4, 7 \) settled as draws by other means ]
In general

$N$ odd and $N \geq 3^d - 1$ \Rightarrow pairing strategy

$N$ even and $N \geq 2^d - 1$

**Proof**

$N$ odd: # lines through any point $(c_1, c_2, \ldots, c_d)$ \leq \frac{3^d - 1}{2}

≤ 3 choices for how line continues in each coordinate
- up, down, constant
- 1 for all constant

Divide by two because each line has two orientations
A even: if point $w = (c_1, c_2, \ldots, c_d)$
and we fix where line is constant $i \in I$
then value $J \leq c_i$. $i \notin I$ determines direction
of line.

E.g. $1 < c_i = x \leq n/2$ and $1 \notin I$
then line must go up.

# choices for $I$ is $2^{d-1}$.

Must also divide by 2.
Quasi-probabilistic Method

If $|A_i| \geq N$ and $N < 2^{n-1}$ then $P_2$ can force a draw.

Proof

At any time in game $C$, in the set $G$ elements that have been colored by player $i$. 
\[ U = A \backslash C_1 \cup C_2 \]

\[ \phi = \sum_{i : A_i \cap C_2 = \emptyset} 2^{-1} |A_i \cap U| \]

= Expected number of sets of color 1 if both players play randomly
P2 keeps $\Phi < 1$ so that at the end, there are no sets of color 1.

After P1 first move

$$\Phi < 2^{n-1} \times \left(\frac{1}{2}\right)^{n-1} < 1$$

If P2 can keep $\Phi < 1$ then the game is drawn since when $U = \emptyset$, $\Phi = \#i : A_i \leq C_1$.
Keep track of $I$ after choices

$x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}, x_k$

$x_i = i^{th}$ choice of $P_1$

$y_i = i^{th}$ choice of $P_2$

$\hat{I}_k = \text{value of } I \text{ here}$
\[ \Phi_{k+1} - \Phi_k = -\sum_{i \in A \cap C} 2^{-|A_i \cap U|} + \sum_{i \in A \cap C, y \in A_i} 2^{-|A_i \cap U|} \]

\[ y \cap A_i \land x_k \in A_i \]

\[ i : A \cap C = \emptyset \]
\[ X \times A = A \cup A \cap \emptyset \]

\[ \forall i : A \cap C = \emptyset \]

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\[ M \]

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So if $P_2$ chooses $y_k$, to maximize

\[ \sum_{i \in A_i \cap C} 2^{-1(A_i \cap U)} \]

\( i \in A_i \cap C \neq \emptyset \)
\( y_k \in A_i \)

then \[ \Phi_{k+1} < \Phi_k < 1 \]