1. Prove that if \( u, v \) are the only vertices of odd degree in a graph \( G \), then there is a path from \( u \) to \( v \) in \( G \).

**Solution:** We have to show that \( u, v \) are in the same component of \( G \). But if they are in different components, \( u \in C_1, v \in C_2 \) then the sub-graph induced by \( C_1 \) has one odd vertex, \( u \). This contradicts the fact that every graph has an even number of vertices.

2. Let \( G = (V, E) \) be a graph with minimum degree at least three. Show that it contains a cycle of even length. (Hint: Consider a longest path).

**Solution:** Let \( P = (x = x_0, x_1, \ldots, x_k) \) be a longest path in \( G \). Let \( x_1, x_i, x_j, 1 < i < j \) be three neighbors of \( x \). If \( i \) is odd then the cycle \((x_0, x_1, \ldots, x_i, x_0)\) has \( i + 1 \) edges and is even and so we can assume that \( i, j \) are both even. But then the cycle \((x_0, x_i, x_{i+1}, \ldots, x_j, x_0)\) has \( j - i + 2 \) edges and is even.

3. Prove that if \( T_1, T_2, \ldots, T_k \) are pair-wise intersecting sub-trees of a tree \( T \), then \( T \) has a vertex common to \( T_1, T_2, \ldots, T_k \). (Hint: use induction on \( k \)).

**Solution:** Assume inductively that \( H = \bigcap_{i=1}^{k} T_i \) is non-empty. \( H \) must be a sub-tree of \( T \), for if \( u, v \in H \) then each \( T_i \) contains the path from \( u \) to \( v \) in \( T \). Now let \( \Gamma = T \setminus H \) be obtained by deleting the vertices of \( H \) from \( T \). Let \( C_1, C_2, \ldots, C_m \) be the components of \( \Gamma \). Each \( C_i \) contains a unique vertex \( v_i \) that is adjacent to \( \Gamma \). If \( C_1 \) contained two such vertices \( u, u' \) then either the path from \( u \) to \( u' \) goes through \( \Gamma \) and then \( u, u' \) are in different components of \( \Gamma \) or it avoids \( \Gamma \) and then \( T \) contains a cycle, contradiction.

Suppose now that \( T_{k+1} \) does not share a vertex with \( \Gamma \). Then \( T_{k+1} \) must be contained in a single component \( C_1 \), say. For if \( T_{k+1} \) meets \( C_1 \) and \( C_2 \) then \( T_{k+1} \) must contain a path from \( C_1 \) to \( C_2 \) and this must go through \( \Gamma \). We claim now that \( v_1 \) belongs to \( T_1, T_2, \ldots, T_{k+1} \). Suppose that \( w \in C_1 \) is in \( T_1 \) and \( T_{k+1} \). Then \( T_1 \) contains a path from \( w \) to \( \Gamma \) and this goes through \( v_1 \). But then \( v_1 \in \Gamma \), contradiction.