1. How many integral solutions of
\[ x_1 + x_2 + x_3 + x_4 + x_5 = 100 \]
satisfy \( x_1 \geq 4, \ x_2 \geq 8, \ x_3 \geq -2, \ x_4 \geq 3 \) and \( x_5 \geq 0 \)?

**Solution** Let
\[ y_1 = x_1 - 4, \quad y_2 = x_2 - 8, \quad y_3 = x_3 + 2, \quad y_4 = x_4 - 3, \quad y_5 = x_5. \]
An integral solution of \( x_1 + x_2 + x_3 + x_4 + x_5 = 100 \) such that \( x_1 \geq 4, \ x_2 \geq 8, \ x_3 \geq -2, \ x_4 \geq 3 \) and \( x_5 \geq 0 \) corresponds to an integral solution of \( y_1 + y_2 + y_3 + y_4 + y_5 = 87 \) such that \( y_1, \ldots, y_5 \geq 0 \). From a result in class,
\[ |\{(y_1, y_2, y_3, y_4, y_5) : y_1, \ldots, y_5 \in \mathbb{Z}_+ \textrm{ and } y_1 + \cdots + y_5 = 87\}| = \binom{87 + 5 - 1}{5 - 1} = \binom{91}{4}. \]

2. Show that if \( n \geq q \geq 0 \) then
\[ \sum_{k=0}^{\ell} \binom{\ell-k}{m} \binom{q+k}{n} = \binom{\ell+q+1}{m+n+1}. \]

**Solution** Let \( S = \binom{\ell+q+1}{m+n+1}. \) If \( \{x_1 < x_2 < \cdots < x_{m+n+1}\} \in S \) then put \( X \in S_k \) if \( x_{m+1} = \ell - k + 1 \). Our assumption \( n \geq q \) implies that \( x_{m+1} \leq \ell + 1 \) and so \( 0 \leq k \leq \ell \). The sets \( S_0, S_1, \ldots, S_{\ell} \) partition \( S \) and \( |S_k| = \binom{\ell-k}{m} \binom{q+k}{n} \).

3. How many ways are there of placing \( k \) 1’s and \( n-k \) 0’s at the vertices of an \( n \) vertex polygon, so that every pair of 1’s are separated by at least \( \ell \) 0’s?

**Solution** Choose a vertex \( v \) of the polygon in \( n \) ways and then place a 1 there. For the remainder we must choose \( a_1, \ldots, a_k \geq \ell \) such that \( a_1 + \cdots + a_k = n - k \) and then go round the cycle (clockwise) putting \( a_1 \) 0’s followed by a 1 and then \( a_2 \) 0’s followed by a 1 etc..

Each pattern \( \pi \) arises \( k \) times in this way. There are \( k \) choices of \( v \) that correspond to a 1 of the pattern. Having chosen \( v \) there is a unique choice of \( a_1, a_2, \ldots, a_k \) that will now give \( \pi \).

There are \( \binom{n-k \ell - 1}{k-1} \) ways of choosing the \( a_i \) and so the answer to our question is
\[ \frac{n}{k} \binom{n-k \ell - 1}{k-1}. \]