# 21-301 Combinatorics, Fall 2009: Test 3

Name: __________________________

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Q1: (40pts)
Prove that if $n > 0$ and $a_1, a_2, \ldots, a_{n+1}$ are distinct positive integers then there is a pair $i < j$ such that $a_i - a_j$ is divisible by $n$.

Solution:
Consider the $n + 1$ integers $u_k \mod n$, $k = 1, 2, \ldots, n + 1$.

By the PHP there exist $k, l$ such that $u_k = u_l$. 

Q2: (40pts)
Let $P_3$ denote the path $a, b, c, d$ with three edges and four distinct vertices.

(a) Show that the edges of $K_4$ can be colored red or blue so that there is neither a red copy of $P_3$ nor a blue copy of $P_3$.

(b) Show that the edges of $K_5$ are colored red or blue then there is either a red copy of $P_3$ or a blue copy of $P_3$.

(Hint: Concentrate first on the possible colorings of the edges of the $4$-cycle $C$ on $\{1, 2, 3, 4\}$.)

Solution:
(a) Color edges 12, 13, 14 red and edges 23, 24, 34 blue.
(b) If red is used 3 or 4 times on $C$ then there is a red $P_3$.

Similarly for blue.
Assume that each color is used twice on $C_4$.

Either
(i) 12, 13 are red and 42, 43 are blue (w.l.o.g.)
If 35 is red then 2135 is red and if 35 is blue then 2435 is blue.

or
(ii) 12, 34 are red and 13, 24 are blue (w.l.o.g.)
If 23 is red then 1234 is red and
if 23 is blue then 1324 is blue.
Q3: (20pts) Re-call that a tournament is a digraph in which for every pair of distinct vertices $u, v$, exactly one of $(u, v), (v, u)$ is an edge. Show that a tournament on $n$ vertices has a directed path with $k$ edges for every $1 \leq k \leq n - 1$.

(Hint: Use induction on $k$.)

**Solution:** The base case $k = 1$ is trivial. 2 points.
Assume there is a path $P = x_1, x_2, \ldots, x_{k+1}$ with $k$ edges where $k < n - 1$. 2 points.
Choose a vertex $y$ not on $P$. 2 points.
If the edge joining $y, x_1$ is oriented $y$ to $x_1$ then $yx_1, x_2, \ldots, x_{k+1}$ is a path with $k + 1$ edges.
Otherwise, if the edge joining $x_{k+1}, y$ is oriented $x_{k+1}$ to $y$ then $x_1, x_2, \ldots, x_{k+1}y$ is a path with $k + 1$ edges.
Otherwise, $y, x_1$ is oriented $x_1$ to $y$ and $x_{k+1}, y$ is oriented $y$ to $x_{k+1}$.
It follows that there exists $1 \leq \ell \leq k$ such that $y, x_{\ell}$ is oriented $x_{\ell}$ to $y$ and $x_{\ell+1}, y$ is oriented $y$ to $x_{\ell+1}$.
Then $x_1, \ldots, x_{\ell}, y, x_{\ell+1}, \ldots, x_{k+1}$ is a path with $k + 1$ edges.