Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2009: Test 1

Name: ___________________________

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Q1: (40pts)

$k$ a’s, $k$ b’s and $n - 2k$ c’s are placed on the vertices of an $n$ vertex polygon so that each $a$ is followed clockwise by $b$ which is followed by a non-empty sequence of c’s until the next $a$. Show that the number of ways of doing this is

$$\frac{n}{k} \binom{n-2k-1}{k-1}.$$  

**Solution:** There are $n$ ways of choosing a place to put an $a$. A $b$ follows immediately. Let there be $x_i$ c’s between the $i$th $b$ and the $i + 1$’th $a$. Then we have $x_1 + \cdots + x_k = n - 2k$ and $x_1, \ldots, x_k \geq 1$. The number of choices for the $x$’s is thus $\binom{n-2k-1}{k-1}$ and we multiply by $n/k$ to account for the choice of the “first” $a$ and for over-counting.
Q2: (40pts)
n children take off their jackets and shoes and put them into a pile on the floor and go and play. How many ways are there of giving to each of the children a pair of matching shoes and a jacket, so that no child gets his/her own jacket and shoes. Your answer should be a summation.

Re-call that if $A_1, A_2, \ldots, A_N \subseteq A$ then

$$
\left| \bigcap_{i=1}^{N} \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.
$$

Solution: Suppse that child $i$ is given the jacket of child $\pi_1(i)$ and the shoes of child $\pi_2(i)$. Let

$$
A_i = \{ (\pi_1, \pi_2) : \pi_1(i) = \pi_2(i) = i \}
$$

for $i = 1, 2, \ldots, n$.

We need to compute $\left| \bigcap_{i=1}^{n} \bar{A}_i \right|$. Now if $|S| = k$ then $|A_S| = ((n-k)!)^2$ since we have fixed $\pi_1(i), \pi_2(i)$ for $i \in S$ and the remaining values can be permuted arbitrarily. Thus

$$
\left| \bigcap_{i=1}^{n} \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} ((n-|S|)!)^2 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} ((n-k)!)^2.
$$
Q3: (20pts) How many ways are there of placing k a’s and n − k b’s on the vertices of an n vertex polygon so that each a is separated by an odd number of b’s. There are different answers for n odd or n even.

Solution: There are n choices as to where to put an a on a vertex. Suppose that after this there are $x_i$ b’s between each a where $x_1 + \cdots + x_k = n - k$ and each $x_i$ is odd. Let $d_k$ be the number of choices for the $x_i$’s, in which case the solution is $nd_k/k$.

Each $x_i$ can be written as $2y_i + 1$ where $y_i \geq 0$ for $i = 1, 2, \ldots, k$ and $2(y_1 + \cdots + y_k) + k = n - k$. If there is a solution then $n = 2(y_1 + \cdots + y_k + k)$ is even i.e. there are no solutions for odd $n$. Otherwise, if $n$ is even then $y_1 + \cdots + y_k = n/2 - k$ and the number of choices for the $y_i$’s and hence the $x_i$’s is $\binom{n/2-k-1}{k-1}$. So the final answer is

$$\frac{n}{k} \binom{n/2 - 1}{k - 1}.$$  \hspace{1cm} (1)

Alternate solution found by some students:
The polygon’s vertices are partitioned into a sequence of segments where each segment starts with an a and continues with an odd number of b’s. Thus each segment is even and so n must be even. Furthermore, if $n = 2m$ and we partition the vertices into $A = \{1, 3, \ldots, 2m - 1\}$ and $B = \{2, 4, \ldots, 2m\}$ then all a’s must be placed in $A$ or all a’s must be placed in $b$ and any such placement is valid. Thus the number of choices is

$$2 \binom{n/2}{k}.$$  

You can check that this is the same as (1).