9/2/09

Principle of Inclusion – Exclusion

PIE
\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]

\(A_1, A_2 \subseteq \mathcal{A}\) and \(\overline{A}_b = A \setminus A_1\)

\[ |\overline{A}_1 \cap \overline{A}_2| = |A| - (|A_1| - |A_2|) + |A_1 \cap A_2| \]

\[ \overrightarrow{\text{De Morgan's Rule.}} \]

\[ \overline{A_1 \cup A_2} \]
\[ |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 | = \\
\quad |\bar{A}| \\
- |A_1| - |A_2| - |A_3| \\
+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\
- |A_1 \cap A_2 \cap A_3| \]
\[ A_1, A_2, \ldots, A_N \subseteq A \]

\[
\left| \bigcap_{i=0}^{N} \overline{A_i} \right| = \\
|A| - |A_1| - |A_2| - \cdots - |A_N| \\
+ |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{N-1} \cap A_N| \\
- |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| - \cdots - |A_{N-2} \cap A_{N-1} \cap A_N| \\
+ |A_1 \cap A_2 \cap A_3 \cap A_4| + \cdots.
\]
\[ S \subseteq \{N\} \quad \text{(nothing to do with } A. \) \]

\[ A_S = \bigcap_{i \in S} A_i \]

\[ = \{ x \in A : x \in A_i \text{ for } i \in S \} \]

\[ A_{\{3,5,12\}} = A_3 \cap A_5 \cap A_{12} \]

\[ A_\emptyset = A \]
\[ \bigcap_{i=1}^{N} \overline{A_i} = \sum_{S \subseteq [n]} (-1)^{|S|} \ell(A_S) \]
Simple Example
How many integers $\leq 10^6$ are not divisible by $5, 8, 12$?

(Not divisible by 5) $A_1 = \{ \text{divisible by 5} \}$

and

(Not divisible by 8) $A_2 = \{ \text{divisible by 8} \}$

and

(Not divisible by 12) $A_3 = \{ \text{divisible by 12} \}$
# = 10^6

- 200,000 - 125,000 - 50,000 \text{, } 1

+ 25,000 + 16,666 + 41,666 \text{, } 2

- 8,333 \text{, } 3
Derangements

A derangement is a permutation \( \pi \) of \( \{1, 2, \ldots, n\} \) such that

\[ \pi(i) \neq i , \quad i=1, 2, \ldots, n \]
\[ n = 10 \]

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
3 & 6 & 5 & 8 & 9 & 7 & 10 & 2 & 1 & 4
\end{array}
\]

A derangement.

\[ \pi \rightarrow \pi(i) \]
Not a derangement

i 1 2 3 4 5 6 7 8
\[\pi(i)\] 2 3 5 4 6 8 7 1

Indicates not a derangement
To use PIE we have to find the $A_i$.

Each $A_i$ is a set of permutations

derangements in (not in $A_1$) and (not in $A_2$) and ___
$A_{i} = \sum_{\pi \in \Pi} \pi(i) = i \uparrow \downarrow$

$\# \text{derangements} =$

$\sum_{S \subseteq [n]} (-1)^{|S|} |A_{S}|$

$|A_{\emptyset}| = n!$
\[ A_i = \pi(i) \quad \forall \pi \in S_n \]

\[ |A_i| = (n-1)! \]

\[ A_{\{3,5,6,\}} \quad \pi(\{i\}) = 1 \times 3 \times 5 \times 6 \]

\[ |A_{\{3,5,6,\}}| = (n-3)! \]
In general

\[ |A_S| = (n-|S|)! \]

\( \# \text{derangements} = \sum_{S \subseteq [n]} (-1)^{|S|} (n-|S|)! = \sum_{k=0}^{n} \sum_{|S|=k} (-1)^k (n-k)! \)

\[ = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \]
\[= n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!} \]

\[\Rightarrow \quad \frac{n!}{e} \quad \text{as} \quad n \to \infty\]