1. Show that the number of sequences out of \( \{a, b, c\}^n \) which do not contain a consecutive sub-sequence of the form \( xx \) where \( x = a, b \) satisfies the recurrence \( b_0 = 1, b_1 = 3 \) and

\[
b_n = b_{n-1} + 2(b_{n-2} + \cdots + b_0) + 2b_0.
\]  

(Hint: Consider the number of sequences where the first \( c \) from the left is at position \( k \).)

Deduce from this that

\[
b_n = 2b_{n-1} + b_{n-2}.
\]  

**Solution:** Let \( B_n \) denote the set of allowed sequences of length \( n \). Suppose that \( c \) appears first in position \( k \). Then the sequence starts \( abab\cdots \) or \( baba\cdots \), then \( c \) and then any sequence from \( B_{n-k} \). Thus the number of such is \( 2b_{n-k} \) for \( 1 \leq k \leq n \). The extra \( 2b_0 = 2 \) counts sequences without \( c \).

To get (2), subtract equation (1) with \( n \) replaced by \( n - 1 \) from equation (1).

2. Show that the number of sequences out of \( \{a, b, c\}^n \) which do not contain a consecutive sub-sequence of the form \( abc \) satisfies the recurrence \( b_0 = 1, b_1 = 3, b_2 = 9 \) and

\[
b_n = 2b_{n-1} + c_n
\]  

\[
c_n = c_{n-1} + b_{n-2} + c_{n-2} + b_{n-3}
\]

where \( c_n \) is the number of such sequences that start with \( a \).

Now find a recurrence only involving \( b_n \), by using (3) to eliminate \( c_n \) from (4).

**Solution:** There are \( 2b_{n-1} \) sequences of the required form that start with \( b \) or \( c \). There are \( c_n \) sequences that start with \( a \). This explains (3).

There are \( c_{n-1} \) sequences that start with \( aa \), \( b_{n-2} \) sequences that start with \( ac \), \( c_{n-2} \) sequences that start with \( aba \) and \( b_{n-3} \) sequences that start with \( abb \). This covers the possibilities for sequences starting with \( a \).

We have

\[
b_n - 2b_{n-1} = b_{n-1} - 2b_{n-2} + b_{n-2} + b_{n-3} - 2b_{n-3} + b_{n-3}
\]

and so

\[
b_n = 3b_{n-1} - b_{n-3}.
\]

3. Let \( a_0, a_1, a_2, \ldots \) be the sequence defined by the recurrence relation

\[
a_n + 3a_{n-1} + 2a_{n-2} = n \quad \text{for } n \geq 2
\]

with initial conditions \( a_0 = 1 \) and \( a_1 = 3 \). Determine the generating function for this sequence, and use the generating function to determine \( a_n \) for all \( n \).
Solution:

\[
\sum_{n=2}^{\infty} (a_n + 3a_{n-1} + 2a_{n-2})x^n = \sum_{n=2}^{\infty} nx^n
\]

\[
a(x) - 1 - 3x + 3x(a(x) - 1) + 2x^2a(x) = \frac{x}{(1-x)^2} - x
\]

\[
a(x)(1 + 3x + 2x^2) = \frac{x}{(1-x)^2} + 1 + 5x
\]

\[
a(x) = \frac{x}{(1+x)(1+2x)(1-x)^2} + \frac{1+5x}{(1+x)(1+2x)}
\]

\[
= \frac{17/4}{1+x} + \frac{-31/9}{1+2x} + \frac{1/36}{1-x} + \frac{1/6}{(1-x)^2}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{17}{4}(-1)^n - \frac{31}{9}(-2)^n + \frac{1}{36} + \frac{1}{6}(n+1) \right) x^n.
\]

So

\[
a_n = \frac{17}{4}(-1)^n - \frac{31}{9}(-2)^n + \frac{1}{36} + \frac{1}{6}(n+1) \quad \text{for } n \geq 0.
\]