1. Use induction to show that
\[
\binom{n-1}{k} = \binom{n}{k} - \binom{n}{k-1} + \cdots \pm \binom{n}{0}.
\]

**Solution** We use induction on \( k \) for a fixed \( n \).

**Base Case:** \( k = 0 \). This is trivial, \( \binom{k}{0} = \binom{k-1}{0} \).

**Inductive Step:** Suppose that the identity is true for some \( k \geq 0 \). Then
\[
\binom{n}{k+1} - \binom{n}{k} + \cdots \pm \binom{n}{0} = \binom{n+1}{k+1} - \binom{n}{k} - \cdots \pm \binom{n}{0}
\]

by induction.

2. (a) Let \( S_k \) denote the collection of \( k \)-sets \( \{1 \leq i_1 < i_2 < \cdots < i_k \leq m-2\} \subseteq [m] \) such that \( i_{t+1} - i_t \geq 2 \) for \( 1 \leq t < k \). Show that
\[
|S_k| = \binom{m-2k}{k}.
\]

(b) How many of the \( 3^n \) sequences \( x_1 x_2 \cdots x_{2n}, x_i \in \{a, b, c\}, i = 1, 2, \ldots, n \) are there such that \( abc \) does not appear as a subsequence e.g. if \( n = 6 \) then we include \( aabbcc \).

**Solution** (a) For a first argument, let \( z_1 = i_1, z_2 = i_2 - i_1, \ldots, z_k = i_k - i_{k+1}, z_{k+1} = m - i_k \). We can count the number of choices for \( z_1, z_2, \ldots, z_{k+1} \). But these are the solutions to
\[
z_1 + z_2 + \cdots + z_{k+1} = m, z_1 \geq 1, z_2, z_3, \ldots, z_k \geq 3, z_{k+1} \geq 2.
\]

The number of such is
\[
\binom{m-1 - 3(k-1) - 2 + k + 1 - 1}{k+1 - 1} = \binom{m-2k}{k}.
\]

Alternatively, we can represent a \( k \)-set by a sequence of \( k \) 1’s and \( m-k \) 0’s in the usual way. Now we need every pair of 1’s separated by at least 2 0’s. We can start with a sequence of \( m-2k \) 0’s, choose \( k \) of them and replace each of these \( k \) 0’s by 100. This process is reversible. For the 0,1 sequences we are counting each 1 is followed by at least 2 0’s. Just replace 100 by 0 to get a sequence of \( m-k \) 0’s.

(b) Let \( A = \{a, b, c\}^n \). Then let
\[
A_k = \{x \in A : x_k = a, x_{k+1} = b, x_{k+2} = c\}
\]
for \( k = 1, 2, \ldots, n - 2 \).

Let \( S = \bigcup_{k \geq 0} S_k \). Then

\[
|A_S| = \begin{cases} 
3^{n-3|S|} & S \in S_{[S]} \\
0 & S \notin S_{[S]} 
\end{cases}
\]

Then we must compute

\[
\left| \bigcap_{i=1}^m \bar{A}_i \right| = \sum_{S \subseteq [m]} (-1)^{|S|} |A_S| \\
= \sum_{S \subseteq S} (-1)^{|S|} 3^{n-3|S|} \\
= 3^n \sum_{k=0}^{m} (-1)^k |S_k| 3^{-3k} \\
= 3^n \sum_{k=0}^{m} (-1)^k \binom{m-2k}{k} 3^{-3k}
\]

3. Find an expression for the size of the set

\[
\{(x_1, x_2 \ldots, x_m) \in \mathbb{Z}^m : x_1 + x_2 + \cdots + x_m = n \text{ and } a \leq x_j \leq b \text{ for } j = 1, 2, \ldots, m\}.
\]

[You should use Inclusion-Exclusion and expect to have your answer as a sum.]

**Solution:** Let

\[
A = \{(x_1, x_2 \ldots, x_m) \in \mathbb{Z}^m : x_1 + x_2 + \cdots + x_m = n \text{ and } a \leq x_j \text{ for } j = 1, 2, \ldots, m\}.
\]

Then let

\[
A_i = \{x \in A : x_i \geq b + 1\}.
\]

Now,

\[
A_S = \{(x_1, x_2 \ldots, x_m) \in \mathbb{Z}^m : x_1 + x_2 + \cdots + x_m = n \text{ and } a \leq x_j \text{ for } j \notin S, b + 1 \leq x_j \text{ for } j \in S\}.
\]

So,

\[
|A_S| = \left( \frac{n - 1 - b|S| - (a - 1)(m - |S|)}{m - 1} \right).
\]

Then we must compute

\[
\left| \bigcap_{i=1}^m \bar{A}_i \right| = \sum_{S \subseteq [m]} (-1)^{|S|} |A_S| \\
= \sum_{S \subseteq [m]} (-1)^{|S|} \left( \frac{n - 1 - b|S| - (a - 1)(m - |S|)}{m - 1} \right) \\
= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left( \frac{n - 1 - bk - (a - 1)(m - k)}{m - 1} \right).
\]