1. Consider the following game: There is a pile of $n$ chips. A move consists of removing any proper factor of $n$ chips from the pile. The player to leave a pile with one chip wins. Determine the $N$ and $P$ positions and a winning strategy from an $N$ position.

**Solution** $n$ is a $P$-position iff it is odd. If $n$ is even then the next player can simply remove one chip. If $n$ is odd, then any factor of $n$ is also odd.

2. Consider the following game: There is a single pile of $n$ chips. A move consists of removing (i) any even number of chips provided it is not the whole pile, or (ii) the whole pile, but only if it has $2 \pmod{3}$ chips. The terminal positions are zero and one. Determine the Sprague-Grundy numbers of each pile size.

(Compute the first 15 numbers and see if you can see a pattern.)

**Solution** The Sprague-Grundy function $g$ is given by

$$g(0) = g(1) = g(4) = 0 \text{ and } g(2) = g(3) = 1 \text{ and } g(k) = \lfloor n/2 \rfloor - 1 \text{ for } n \geq 5.$$

We verify the last claim by induction. It can be checked for $n = 5, 6$. Suppose next that $k > 3$. Then if $*$ is $g(0) = 0$ for $n \mod 3 = 2$ and not there otherwise,

$$g(2k) = \text{mex}\{g(2k - 2), g(2k - 4), \ldots, g(6), g(4), g(2), *\}$$
$$= \text{mex}\{k - 2, k - 3, \ldots, 2, 0, 1, *\}$$
$$= k - 1.$$

$$g(2k + 1) = \text{mex}\{g(2k - 1), g(2k - 3), \ldots, g(5), g(3), g(1), *\}$$
$$= \text{mex}\{k - 2, k - 3, \ldots, 2, 1, 0, *\}$$
$$= k - 1.$$

3. Consider the following multi-pile game. A move consists of either (i) removing one, two or three chips from any pile, or (ii) splitting a pile of
size \( n \geq 2 \) into two piles of sizes 1 and \( n - 1 \). Determine the Sprague-Grundy numbers for a game that starts with a single pile of size \( n \). (Compute the first 15 numbers and see if you can see a pattern.)

Suppose that the current position consists of three piles of sizes 3, 5 and 7. Show that this is an N-position and find all of the winning moves.

**Solution**

\[
g(i) = i, \; i = 0, 1, 2 \text{ and } g(3) = 4. \tag{1}
\]

\[
g(4k) = 0, \; g(4k + 1) = 3, \; g(4k + 2) = 1, \; g(4k + 3) = 2 \quad \text{for } k \geq 1. \tag{2}
\]

To verify (1) remember that moves from \( n = 3 \) yield 0, 1, 2 or 1\( + 2 \) and \( g(1 + 2) = g(1) \oplus g(2) \). So \( g(3) = \text{mex}\{g(0), g(1), g(2), g(1) \oplus g(2)\} = \text{mex}\{0, 1, 2, 1 \oplus 2\} = \text{mex}\{0, 1, 2, 3\} = 4 \).

We verify (2) by induction. The case \( k = 1 \) is done case by case:

\[
g(4) = \text{mex}\{g(1), g(2), g(3), g(3) \oplus g(1)\}
= \text{mex}\{1, 2, 3, 4 \oplus 1\} = \text{mex}\{1, 2, 3, 5\} = 0.
\]

\[
g(5) = \text{mex}\{g(2), g(3), g(4), g(4) \oplus g(1)\}
= \text{mex}\{2, 4, 0, 0 \oplus 1\} = \text{mex}\{2, 4, 0, 1\} = 3.
\]

\[
g(6) = \text{mex}\{g(3), g(4), g(5), g(5) \oplus g(1)\}
= \text{mex}\{4, 0, 3, 3 \oplus 1\} = \text{mex}\{4, 0, 3, 2\} = 1.
\]

\[
g(7) = \text{mex}\{g(4), g(5), g(6), g(6) \oplus g(1)\}
= \text{mex}\{0, 3, 1, 1 \oplus 1\} = \text{mex}\{0, 3, 1, 0\} = 2.
\]

We verify (2) by induction on \( k \):

\[
g(4k) = \text{mex}\{g(4k - 1), g(4k - 2), g(4k - 3), g(4k - 1) \oplus 1\}
= \text{mex}\{2, 1, 3, 2 \oplus 1\} = \text{mex}\{2, 1, 3, 3\} = 0.
\]

\[
g(4k + 1) = \text{mex}\{g(4k), g(4k - 1), g(4k - 2), g(4k) \oplus 1\}
= \text{mex}\{0, 2, 1, 0 \oplus 1\} = \text{mex}\{0, 2, 1, 1\} = 3.
\]
\[ g(4k + 2) = \text{mex}\{g(4k + 1), g(4k), g(4k - 1), g(4k + 1) \oplus 1\} \\
= \text{mex}\{3, 0, 2, 3 \oplus 1\} = \text{mex}\{3, 0, 2, 2\} = 1. \]

\[ g(4k + 3) = \text{mex}\{g(4k + 2), g(4k + 1), g(4k), g(4k + 2) \oplus 1\} \\
= \text{mex}\{0, 3, 1, 0 \oplus 1\} = \text{mex}\{0, 3, 1, 1\} = 2. \]

If the current position is 3,5,7 then is value is \[ g(3, 5, 7) = g(3) \oplus g(5) \oplus g(7) = 4 \oplus 3 \oplus 2 = 5 \neq 0 \] and so it is an N-position.

The unique winning move is to take 2 chips from the 3-pile. The new position has value \[ g(1) \oplus g(5) \oplus g(7) = 1 \oplus 3 \oplus 2 = 0. \]