1. Prove that if we 2-color the edges of $K_n$ then either (i) there is a vertex of Red degree at least $\frac{n}{2} - 1$ or (ii) there is a Blue triangle. Show also that it is possible to have a 2-coloring in which the maximum Red degree is $\frac{n}{2} - 1$ and in which there is no Blue triangle.

**Solution** If there is no vertex of Red degree $\geq \frac{n}{2} - 1$ then every vertex has minimum Blue degree $\geq \frac{n}{2} + 1$. Thus the number of Blue edges is greater than $\frac{n^2}{4}$ and so by Turan’s Theorem (Graph Theory Notes p60) there is a Blue triangle.

**Alternative proof** Let $(u, v)$ be a Blue edge. Both $u, v$ have at least $\frac{n}{2}$ Blue neighbors outside $u, v$. This means they have a common Blue neighbor.

If $n = 2m$ we can split the vertex set $[n]$ into two sets $A, B$ of size $m$. Then if we color edges inside $A$ or inside $B$ Red and edges between $A, B$ Blue then every vertex has Red degree $\frac{n}{2} - 1$ and there is no Blue triangle.

2. Prove that if we 2-color the edges of $K_6$ then there are at least two monochromatic triangles.

**Solution** Assume w.l.o.g. that triangle $(1, 2, 3)$ is Red and that $(4, 5, 6)$ is not Red and in particular that edge $(4, 5)$ is Blue. If $x = 4, 5$ or 6 then there can be at most one Red edge joining $x$ to 1, 2, 3, else we get a Red triangle. So we can assume that there are two Blue edges joining each of 4, 5 to 1, 2, 3. So there must be $x \in \{1, 2, 3\}$ such that both $(x, 4)$ and $(x, 5)$ are Blue. But then triangle $(x, 4, 5)$ is Blue.

**Alternative proof** Given the coloring, let us count the number $N$ of paths of length two which consist of a Red edge followed by a Blue edge. Let $r_i$ denote the number of Red edges incident with $i$. Then we have

$$N = \sum_{i=1}^{6} r_i(6 - r_i) \leq \sum_{i=1}^{6} 6 = 36.$$ 

Each of the 20 triangles of $K_6$ contains 0 or 2 of these paths. So at most 18 contain 2 and there are at least 2 mono-colored triangles.
3. Prove that if \( n \geq R(2k, 2k) \) and if we 2-color the edges of \( K_{n,n} \) then there is a mono-chromatic copy of \( K_{k,k} \).

**Solution** Given a coloring \( \sigma \) of \( K_{n,n} \) we construct a coloring \( \tau \) of the edges of \( K_n \) as follows. If \( i < j \) then we give the edge \((i, j)\) of \( K_n \) the same color that is given to edge \((i, j)\) under \( \sigma \).

Since \( n \geq R(2k, 2k) \) we see that \( K_n \) contains a mono-colored copy of \( K_{2k} \). If the set of vertices of this copy is \( S \), divide \( S \) into two parts \( S_1, S_2 \) of size \( k \) where \( \max S_1 < \min S_2 \). It follows that the bipartite sub-graph of \( K_{n,n} \) defined by \( S_1, S_2 \) is mono-colored under \( \sigma \).