II. Sequences

By a real sequence we mean a function \( x : \mathbb{N} \to \mathbb{R} \), i.e. a function whose domain is the set of natural numbers and whose values are real numbers. For each \( n \in \mathbb{N} \) the function value \( x(n) \) is called the \( n \)th term of the sequence. It is customary to write \( x_n \) in place of \( x(n) \) and to denote the sequence by \( \{x_n\}^\infty_{n=1} \). Although we will generally adopt the customary notation, it is important to bear in mind that a sequence is a function. Throughout this section we use the term sequence to mean real sequence. Most of our effort with sequences will be devoted to understanding how the terms \( x_n \) behave when the index \( n \) is large.

The central notion pertaining to sequences is that of a limit. Let \( \{x_n\}^\infty_{n=1} \) be a sequence and \( l \in \mathbb{R} \) be given. We say that \( l \) is a limit of \( \{x_n\}^\infty_{n=1} \) and we write \( x_n \to l \) as \( n \to \infty \) provided that for every \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_n - l| < \epsilon \) for all \( n \in \mathbb{N} \) with \( n \geq N \). A sequence can have at most one limit. (See Proposition II.1.) Therefore, if \( x_n \to l \) as \( n \to \infty \), we refer to \( l \) as the limit of the sequence and we write \( \lim_{n \to \infty} x_n = l \).

A. Some Definitions

Let \( \{x_n\}^\infty_{n=1} \) be a sequence.

**Definition 1:** We say that \( \{x_n\}^\infty_{n=1} \) is convergent if there exists \( l \in \mathbb{R} \) such that \( x_n \to l \) as \( n \to \infty \).

**Definition 2:** We say that \( \{x_n\}^\infty_{n=1} \) is

(i) bounded below if there exists \( \alpha \in \mathbb{R} \) such that \( x_n \geq \alpha \) for all \( n \in \mathbb{N} \).

(ii) bounded above if there exists \( \beta \in \mathbb{R} \) such that \( x_n \leq \beta \) for all \( n \in \mathbb{N} \).

(iii) bounded if there exists \( M \in \mathbb{R} \) such that \( |x_n| \leq M \) for all \( n \in \mathbb{N} \).

**Definition 3:** We say that \( \{x_n\}^\infty_{n=1} \) is

(i) increasing if \( x_{n+1} \geq x_n \) for all \( n \in \mathbb{N} \).

(ii) strictly increasing if \( x_{n+1} > x_n \) for all \( n \in \mathbb{N} \).

(iii) decreasing if \( x_{n+1} \leq x_n \) for all \( n \in \mathbb{N} \).

(iv) strictly decreasing if \( x_{n+1} < x_n \) for all \( n \in \mathbb{N} \).

(v) monotonic if it is either increasing or decreasing.

(vi) strictly monotonic if it is either strictly increasing or strictly decreasing.

**Definition 4:** We say that \( \{x_n\}^\infty_{n=1} \) is a Cauchy sequence provided that for every \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_m - x_n| < \epsilon \) for all \( m, n \in \mathbb{N} \) with \( m, n \geq N \).

**Definition 5:** By a subsequence of \( \{x_n\}^\infty_{n=1} \) we mean a sequence of the form \( \{x_{n_k}\}^\infty_{k=1} \) where \( \{n_k\}^\infty_{k=1} \) is a strictly increasing sequence of natural numbers.
**Definition 6:** Let \( l \in \mathbb{R} \) be given. We say that \( l \) is a cluster point of \( \{x_n\}_{n=1}^{\infty} \) provided that for every \( \epsilon > 0 \), \( \{n \in \mathbb{N} : |x_n - l| < \epsilon\} \) is infinite.

**Definition 7:** Assume that \( \{x_n\}_{n=1}^{\infty} \) is bounded. For each \( n \in \mathbb{N} \) put

\[
y_n = \inf \{x_k : k \in \mathbb{N}, k \geq n\},
\]

\[
z_n = \sup \{x_k : k \in \mathbb{N}, k \geq n\}.
\]

Note that \( \{y_n\}_{n=1}^{\infty} \) is increasing and bounded above and that \( \{z_n\}_{n=1}^{\infty} \) is decreasing and bounded below. We define

\[
\lim \inf_{n \to \infty} x_n = \lim_{n \to \infty} y_n \quad \text{and} \quad \lim \sup_{n \to \infty} x_n = \lim_{n \to \infty} z_n.
\]

(Note that \( \{y_n\}_{n=1}^{\infty} \) and \( \{z_n\}_{n=1}^{\infty} \) are convergent by virtue of Theorem II.6.)

**B. Some Key Results**

**II.1 Proposition:** A sequence can have at most one limit.

**II.2 Proposition:** Every convergent sequence is bounded.

**II.3 Proposition:** Let \( \ell, L, \alpha \in \mathbb{R} \) be given and \( \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \) be sequences. Assume that \( x_n \to \ell \) and \( y_n \to L \) as \( n \to \infty \). Then:

(i) \( x_n + y_n \to \ell + L \) as \( n \to \infty \);

(ii) \( \alpha x_n \to \alpha \ell \) as \( n \to \infty \);

(iii) \( x_n y_n \to \ell L \) as \( n \to \infty \);

(iv) If \( x_n \neq 0 \) for all \( n \in \mathbb{N} \) and \( \ell \neq 0 \), we have \( \frac{1}{x_n} \to \frac{1}{\ell} \) as \( n \to \infty \).

**II.4 Proposition:** Let \( \ell, L \in \mathbb{R} \) be given and \( \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \) be sequences. If \( x_n \leq y_n \) for all \( n \in \mathbb{N} \) and \( x_n \to \ell, y_n \to L \) as \( n \to \infty \) then \( \ell \leq L \).

**II.5 Squeeze Theorem:** Let \( \ell \in \mathbb{R} \) be given and \( \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \) be sequences. Assume that \( x_n \leq y_n \leq z_n \) for all \( n \in \mathbb{N} \) and that \( x_n \to \ell, z_n \to \ell \) as \( n \to \infty \). Then \( y_n \to \ell \) as \( n \to \infty \).

**II.6 Theorem:** Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence.

(i) If \( \{x_n\}_{n=1}^{\infty} \) is increasing and bounded above then \( \{x_n\}_{n=1}^{\infty} \) is convergent.

(ii) If \( \{x_n\}_{n=1}^{\infty} \) is decreasing and bounded below then \( \{x_n\}_{n=1}^{\infty} \) is convergent.

**II.7 Proposition:** Let \( \ell \in \mathbb{R} \) be given and \( \{x_n\}_{n=1}^{\infty} \) be a sequence. Then \( \ell \) is a cluster point of \( \{x_n\}_{n=1}^{\infty} \) if and only if there is a subsequence \( \{x_{n_k}\}_{k=1}^{\infty} \) such that \( x_{n_k} \to \ell \) as \( k \to \infty \).

**II.8 Proposition:** Let \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) be bounded sequences and \( \alpha \in \mathbb{R} \) be given. Then:
(i) \( \limsup_{n \to \infty} (x_n + y_n) \leq \left( \limsup_{n \to \infty} x_n \right) + \left( \limsup_{n \to \infty} y_n \right) \);
(ii) \( \liminf_{n \to \infty} (x_n + y_n) \geq \left( \liminf_{n \to \infty} x_n \right) + \left( \liminf_{n \to \infty} y_n \right) \);
(iii) If \( \alpha \geq 0 \) we have \( \limsup_{n \to \infty} (\alpha x_n) = \alpha \limsup_{n \to \infty} x_n \) and \( \liminf_{n \to \infty} (\alpha x_n) = \alpha \liminf_{n \to \infty} x_n \);
(iv) \( \limsup_{n \to \infty} (-x_n) = -\liminf_{n \to \infty} x_n \) and \( \liminf_{n \to \infty} (-x_n) = -\limsup_{n \to \infty} x_n \).

II.9 Lemma: Let \( \{x_n\}_{n=1}^\infty \) be a bounded sequence and \( l_s \in \mathbb{R} \) be given. Then \( l_s = \limsup_{n \to \infty} x_n \) if and only if (i) and (ii) below hold.

(i) \( \forall \epsilon > 0, \exists N \in \mathbb{N} \) such that \( x_n < l_s + \epsilon \) for all \( n \in \mathbb{N} \) with \( n \geq N \).
(ii) \( \forall \epsilon > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N} \) with \( n \geq N \) such that \( x_n > l_s - \epsilon \).

II.10 Proposition: Let \( \{x_n\}_{n=1}^\infty \) be a bounded sequence. Then \( \limsup_{n \to \infty} x_n \) is the largest cluster point of \( \{x_n\}_{n=1}^\infty \) and \( \liminf_{n \to \infty} x_n \) is the smallest cluster point of \( \{x_n\}_{n=1}^\infty \).

II.11 Proposition: Let \( \{x_n\}_{n=1}^\infty \) be a bounded sequence and put \( l_i = \liminf_{n \to \infty} x_n \) and \( l_s = \limsup_{n \to \infty} x_n \). Let \( \epsilon > 0 \) be given. Then there exists \( N \in \mathbb{N} \) such that

\[ l_i - \epsilon < x_n < l_s + \epsilon \]

for all \( n \in \mathbb{N} \) with \( n \geq N \).

II.12 Proposition: Let \( \{x_n\}_{n=1}^\infty \) be a bounded sequence. Then \( \{x_n\}_{n=1}^\infty \) is convergent if and only if

\[ \limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n. \]

II.13 Bolzano-Weierstrass Theorem: Every bounded sequence has a convergent subsequence.

II.14 Theorem (Cauchy’s Criterion): A sequence is convergent if and only if it is a Cauchy sequence.

II.15 Lemma: Every sequence has a monotonic subsequence.

C. Some Remarks.

II.16 Remark: Let \( \{x_n\}_{n=1}^\infty \) be a sequence. Then \( \{x_n\}_{n=1}^\infty \) is

(i) increasing if and only if \( x_m \geq x_n \) for all \( m, n \in \mathbb{N} \) with \( m \geq n \).
(ii) strictly increasing if and only if \( x_m > x_n \) for all \( m, n \in \mathbb{N} \) with \( m > n \).
(iii) decreasing if and only if \( x_m \leq x_n \) for all \( m, n \in \mathbb{N} \) with \( m \geq n \).
(iv) strictly decreasing if and only if \( x_m < x_n \) for all \( m, n \in \mathbb{N} \) with \( m > n \).

II.17 Remark: Let \( \{n_k\}_{k=1}^\infty \) be a strictly increasing sequence of natural numbers. Then \( n_k \geq k \) for all \( k \in \mathbb{N} \).
II.18 Remark: Let $K$ be an infinite subset of $\mathbb{N}$. Then there is exactly one strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $\{n_k : k \in \mathbb{N}\} = K$.

II.19 Remark: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and $l \in \mathbb{R}$ be given. Then $l$ is a cluster point of $\{x_n\}$ if and only if for every $\epsilon > 0$ and every $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with $n \geq N$ such that $|x_n - l| < \epsilon$.

D. Some Proofs.

Proof of II.1: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and let $l, L \in \mathbb{R}$ be given. Suppose that $x_n \to l$ as $n \to \infty$ and that $x_n \to L$ as $n \to \infty$. We shall show that $L = l$. Let $\epsilon > 0$ be given. Choose $N_1, N_2 \in \mathbb{N}$ such that

1. $|x_n - l| < \epsilon$ for all $n \in \mathbb{N}$, $n \geq N_1$,

2. $|x_n - L| < \epsilon$ for all $n \in \mathbb{N}$, $n \geq N_2$.

Put $N = \max\{N_1, N_2\}$ and notice that

3. $|x_N - l| < \epsilon$, $|x_N - L| < \epsilon$.

Now we observe that

4. $l - L = l - x_N + x_N - L$

and consequently

5. $|l - L| \leq |l - x_N| + |x_N - L| < \epsilon + \epsilon = 2\epsilon$

by virtue of the triangle inequality and (3). Since $\epsilon > 0$ was arbitrary, it follows from (5) that $l - L = 0$. [Indeed, if $l - L \neq 0$ then we may put $\epsilon = \frac{1}{2}|l - L|$ in (5) which yields $|l - L| < |l - L|$ and this is impossible.]

Proof of II.2: Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence and put $l = \lim_{n \to \infty} x_n$. Using the definition of limit with $\epsilon = 1$, we choose $N \in \mathbb{N}$ such that

6. $|x_n - l| < 1$ for all $n \in \mathbb{N}$, $n \geq N$.

Let $S = \{|x_1|, |x_2|, \ldots, |x_N|\}$. Since $S$ is nonempty and finite, it has a largest element. Let $K = \max (S)$ and $M = \max \{1 + |l|, K\}$. Let $n \in \mathbb{N}$ be given. If $n \leq N$ then $|x_n| \in S$ so that
\( |x_n| \leq K \leq M. \)

If \( n \geq N \), then we have

\( x_n = l + x_n - l \)

which yields

\( |x_n| \leq |l| + |x_n - l| \leq |l| + 1 \leq M \)

by virtue of the triangle inequality, (6), and the definition of \( M \). We conclude that 
\( |x_n| \leq M \) for all \( n \in \mathbb{N} \), i.e. \( \{x_n\}_{n=1}^{\infty} \) is bounded.

Proof of II.3 (i): Let \( \epsilon > 0 \) be given. Choose \( N_1, N_2 \in \mathbb{N} \) such that

\( |x_n - l| < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N}, \ n \geq N_1, \)

\( |y_n - L| < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N}, \ n \geq N_2 \)

and put \( N = \max\{N_1, N_2\} \). Then for all \( n \in \mathbb{N} \) with \( n \geq N \) we have

\( |x_n + y_n - (l + L)| = |(x_n - l) + (y_n - L)| \)

\( \leq |x_n - l| + |y_n - L| \)

\( < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \)

by virtue of the triangle inequality and (10), (11).

Proof of II.3 (iii): Since \( \{x_n\}_{n=1}^{\infty} \) is convergent we may choose \( M > 0 \) such that

\( |x_n| \leq M \quad \forall n \in \mathbb{N}. \)

Let \( \epsilon > 0 \) be given. Choose \( N_1, N_2 \in \mathbb{N} \) such that

\( |x_n - l| < \frac{\epsilon}{2(|L| + 1)} \quad \forall n \in \mathbb{N}, \ n \geq N_1, \)

\( |y_n - L| < \frac{\epsilon}{2M} \quad \forall n \in \mathbb{N}, \ n \geq N_2. \)
Put $N = \max\{N_1, N_2\}$. Then, for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$
|x_n y_n - L| = |x_n y_n - L x_n + L x_n - l L|
$$

$$
= |x_n(y_n - L) + L(x_n - l)|
$$

$$
\leq |x_n| \cdot |y_n - L| + |L| \cdot |x_n - l|
$$

$$
< M \left( \frac{\epsilon}{2M} \right) + \frac{|L|\epsilon}{2(|L| + 1)} < \epsilon
$$

by virtue of (13), (14), (15). $\blacksquare$

**Proof of II.4**: Assume that $x_n \leq y_n$ for all $n \in \mathbb{N}$ and that $x_n \to l$, $y_n \to L$ as $n \to \infty$. Put

$$
z_n = y_n - x_n \quad \forall n \in \mathbb{N},
$$

$$
\alpha = L - l
$$

and notice that $z_n \geq 0$ for all $n \in \mathbb{N}$ and that $z_n \to \alpha$ as $n \to \infty$. We shall show that $\alpha \geq 0$, which yields $l \leq L$.

Suppose that $\alpha < 0$. Then we may choose $N \in \mathbb{N}$ such that

$$
|z_n - \alpha| < -\frac{\alpha}{2} \quad \forall n \in \mathbb{N}, n \geq N, \text{ i.e.}
$$

$$
\frac{\alpha}{2} < z_n - \alpha < -\frac{\alpha}{2} \quad \forall n \in \mathbb{N}, n \geq N.
$$

It follows from (20) that

$$
\frac{\alpha}{2} < z_N < -\frac{\alpha}{2} < 0
$$

and this is a contradiction (since $z_n \geq 0$ for all $n \in \mathbb{N}$). We therefore conclude that $\alpha \geq 0$ and hence that $l \leq L$. $\blacksquare$

**Proof of II.6 (i)**: Assume that $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above. Put $S = \{x_n : n \in \mathbb{N}\}$ and observe that $S$ is nonempty and bounded above. Let

$$
l = \sup(S).
$$
We shall show that \( x_n \to l \) as \( n \to \infty \). Let \( \epsilon > 0 \) be given. Then \( l - \epsilon \) is not an upper bound for \( S \). We may therefore choose \( N \in \mathbb{N} \) such that

\[
(23) \quad x_N > l - \epsilon
\]

Recall that

\[
(24) \quad x_n \leq l \quad \forall n \in \mathbb{N}.
\]

Since \( \{x_n\}_{n=1}^{\infty} \) is increasing we deduce from (23) and (24) that

\[
(25) \quad l - \epsilon < x_N \leq x_n \leq l \quad \forall n \in \mathbb{N}, \; n \geq N.
\]

It follows from (25) that

\[
(26) \quad |x_n - l| < \epsilon \quad \forall n \in \mathbb{N}, \; n \geq N. \quad \Box
\]

**Proof of II.9:** For each \( n \in \mathbb{N} \), put

\[
(27) \quad T_n = \{x_k : k \in \mathbb{N}, \; k \geq n\},
\]

\[
(28) \quad z_n = \sup(T_n).
\]

Recall that \( \{z_n\}_{n=1}^{\infty} \) is decreasing and that

\[
(29) \quad \lim_{n \to \infty} z_n = \limsup_{n \to \infty} x_n.
\]

Assume first that \( l_s = \limsup_{n \to \infty} x_n \). We shall show that (i) and (ii) hold. Let \( \epsilon > 0 \) be given. Choose \( N \in \mathbb{N} \) such that

\[
(30) \quad |z_n - l_s| < \epsilon \quad \forall n \in \mathbb{N}, \; n \geq N.
\]

Then, for all \( n \in \mathbb{N} \) with \( n \geq N \) we have

\[
(31) \quad z_n - l_s < \epsilon, \; \text{i.e.}
\]

\[
(32) \quad z_n < l_s + \epsilon,
\]
which yields

\[(33) \quad x_n \leq z_n < l_s + \epsilon\]

and consequently (i) holds. To verify (ii), let \(\epsilon > 0\) and \(N \in \mathbb{N}\) be given. Since \(\{z_n\}_{n=1}^{\infty}\) is decreasing and \(z_n \to l_s\) as \(n \to \infty\), we know that

\[(34) \quad z_n \geq l_s > l_s - \epsilon \quad \forall n \in \mathbb{N}.\]

It follows from (34) that \(l - \epsilon\) is not an upper bound for \(T_N\). We may therefore choose \(y \in T_N\) with \(y > l_s - \epsilon\). By the definition of \(T_N\), \(y = x_n\) for some \(n \in \mathbb{N}\) with \(n \geq N\).

Conversely, assume now that (i) and (ii) hold. We shall show that \(l_s = \limsup_{n \to \infty} x_n\).

Let \(\epsilon > 0\) be given. It follows from (ii) that

\[(35) \quad z_n > l_s - \epsilon \quad \forall n \in \mathbb{N}.\]

Using (i), we choose \(N \in \mathbb{N}\) such that

\[(36) \quad x_n < l_s + \frac{\epsilon}{2} \quad \forall n \in \mathbb{N}, \quad n \geq N.\]

It follows from (36) that

\[(37) \quad z_n \leq l_s + \frac{\epsilon}{2} < l_s + \epsilon.\]

Since \(\{z_n\}_{n=1}^{\infty}\) is decreasing, (37) yields

\[(38) \quad z_n < l_s + \epsilon \quad \forall n \in \mathbb{N}, \quad n \geq N.\]

Combining (35) and (38) we arrive at

\[(39) \quad |z_n - l_s| < \epsilon \quad \forall n \in \mathbb{N}, \quad n \geq N.\]

We conclude that \(z_n \to l_s\) as \(n \to \infty\) and consequently \(l_s = \limsup_{n \to \infty} x_n\). \(\blacksquare\)

**Proof of II.12:** For each \(n \in \mathbb{N}\), put

\[(40) \quad T_n = \{x_k : k \in \mathbb{N}, \quad k \geq n\},\]

\[(41) \quad y_n = \inf(T_n),\]
Let \( l \in \mathbb{R} \) be given. Assume first that \( \limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = l \). We shall show that \( x_n \to l \) as \( n \to \infty \). Observe that

\[
(43) \quad y_n \leq x_n \leq z_n \quad \forall n \in \mathbb{N}.
\]

Since \( y_n \to l \) and \( z_n \to l \) as \( n \to \infty \), it follows from the Squeeze Theorem that \( x_n \to l \) as \( n \to \infty \).

Assume now that \( x_n \to l \) as \( n \to \infty \). We shall show that \( y_n \to l \) and \( z_n \to l \) as \( n \to \infty \). Let \( \epsilon > 0 \) be given. Choose \( N \in \mathbb{N} \) such that

\[
(44) \quad |x_n - l| < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N}, \ n \geq N, \ i.e.
\]

\[
(45) \quad -\frac{\epsilon}{2} < x_n - l < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N}, \ n \geq N.
\]

It follows from (45) that

\[
(46) \quad l - \frac{\epsilon}{2} < x_n < l + \frac{\epsilon}{2} \quad \forall n \in \mathbb{N}, \ n \geq N.
\]

Using (46) we conclude that \( l - \frac{\epsilon}{2} \) is a lower bound for \( T_N \) and \( l + \frac{\epsilon}{2} \) is an upper bound for \( T_N \). It therefore follows that

\[
(47) \quad y_N \geq l - \frac{\epsilon}{2}
\]

\[
(48) \quad z_N \leq l + \frac{\epsilon}{2}.
\]

Since \( \{y_n\}_{n=1}^{\infty} \) is increasing and \( \{z_n\}_{n=1}^{\infty} \) is decreasing we infer from (47), (48) that

\[
(49) \quad l - \frac{\epsilon}{2} \leq y_N \leq y_n \leq z_n \leq z_N \leq l + \frac{\epsilon}{2} \quad \forall n \in \mathbb{N}, \ n \geq N.
\]

It follows immediately from (49) that

\[
(50) \quad |y_n - l| \leq \frac{\epsilon}{2} < \epsilon \quad \forall n \in \mathbb{N}, \ n \geq N,
\]
Let \( \{x_n\}_{n=1}^{\infty} \) be a bounded sequence and put \( l_s = \limsup_{n \to \infty} x_n \). It follows easily from Lemma II.9 that \( l_s \) is a cluster point of \( \{x_n\}_{n=1}^{\infty} \). By Proposition II.7, there is a subsequence \( \{x_{n_k}\}_{k=1}^{\infty} \) of \( \{x_n\}_{n=1}^{\infty} \) such that \( x_{n_k} \to l_s \) as \( n \to \infty \).

Proof of II.14: Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence. Assume first that \( \{x_n\}_{n=1}^{\infty} \) is convergent and put \( l = \lim_{n \to \infty} x_n \). Let \( \epsilon > 0 \) be given and choose \( n \in \mathbb{N} \) such that

\[
|z_n - l| < \frac{\epsilon}{2} < \epsilon \quad \forall n \in \mathbb{N}, \; n \geq N; \tag{51}
\]

i.e. \( y_n \to l \) and \( z_n \to l \) as \( n \to \infty \).

Proof of II.13: Let \( \{x_n\}_{n=1}^{\infty} \) be a bounded sequence and put \( l_s = \limsup_{n \to \infty} x_n \). It follows easily from Lemma II.9 that \( l_s \) is a cluster point of \( \{x_n\}_{n=1}^{\infty} \). By Proposition II.7, there is a subsequence \( \{x_{n_k}\}_{k=1}^{\infty} \) of \( \{x_n\}_{n=1}^{\infty} \) such that \( x_{n_k} \to l_s \) as \( n \to \infty \).

Observe that for all \( m, n \in \mathbb{N} \) with \( m, n \geq N \) we have

\[
x_m - x_n = x_m - l + l - x_n, \tag{53}
\]

which yields

\[
|x_m - x_n| \leq |x_m - l| + |l - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{54}
\]

by virtue of the triangle inequality and (54).

Assume now that \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence. We shall first show that \( \{x_n\}_{n=1}^{\infty} \) is bounded. For this purpose, we choose \( N^* \in \mathbb{N} \) such that

\[
|x_m - x_n| < 1 \quad \forall m, n \in \mathbb{N}, \; m, n \geq N^*; \tag{55}
\]

Put \( S = \{|x_1|, |x_2|, \ldots, |x_{N^*}|\} \) and let \( K = \max (S) \). Then, put \( M = \max \{K, |x_{N^*}| + 1\} \). Let \( n \in \mathbb{N} \) be given. If \( n \leq N^* \) then

\[
|x_n| \leq K \leq M. \tag{56}
\]

If \( n \geq N^* \) then

\[
|x_n| \leq |x_n - x_{N^*}| + |x_{N^*}| < 1 + |x_{N^*}| \leq M. \tag{57}
\]

We conclude that \( |x_n| \leq M \) for all \( n \in \mathbb{N} \), i.e. \( \{x_n\}_{n=1}^{\infty} \) is bounded.

By the Bolzano-Weierstrass Theorem we may choose a convergent subsequence \( \{x_{n_k}\}_{k=1}^{\infty} \). Let \( l = \lim_{k \to \infty} x_{n_k} \). We shall show that \( x_n \to l \) as \( n \to \infty \). Let \( \epsilon > 0 \) be given. Choose \( K, N \in \mathbb{N} \) such that
\[(58)\]
\[|x_{n_k} - l| < \frac{\epsilon}{2} \quad \forall k \in \mathbb{N}, \ k \geq K\]

\[(59)\]
\[|x_m - x_n| < \frac{\epsilon}{2} \quad \forall m, n \in \mathbb{N}, \ m, n \geq N.\]

We choose \(k^* \in \mathbb{N}\) such that \(k^* \geq K\) and \(n_{k^*} \geq N\). (Notice that \(k^* = \max \{K, N\}\) will do.) Then, for all \(n \in \mathbb{N}\) with \(n \geq N\) we have

\[(60)\]
\[x_n - l = x_n - x_{n_{k^*}} + x_{n_{k^*}} - l,\]

which gives

\[(61)\]
\[|x_n - l| \leq |x_n - x_{n_{k^*}}| + |x_{n_{k^*}} - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\]

by virtue of (58), (59), and the triangle inequality. ■