1. Let $A$ and $B$ be events having positive probability. State whether each of the following statements is (i) necessarily true, (ii) necessarily false, or (iii) possibly true.

(a) If $A$ and $B$ are mutually exclusive, then they are independent. (ii)
(b) If $A$ and $B$ are independent, then they are mutually exclusive. (ii)
(c) $P(A) = P(B) = 0.6$, and $A$ and $B$ are mutually exclusive. (ii)
(d) $P(A) = P(B) = 0.6$, and $A$ and $B$ are independent. (iii)

2. In how many ways can 8 people be seated in a row if

(i) there are no restrictions on the seating arrangement;

Solution. 8!

(ii) persons A and B must sit next to each other;

Solution. Treat A and B together as one person. There are 7! ways to seat. There are 2! ways to arrange A and B. Then totally there are 2!7! ways.

(iii) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other;

Solution. Two cases. One is MWMWMWMW, the other is WMWMWMWM. For the first case, the number of ways to sit 4 men is 4! and the number of ways to sit 4 women is also 4!. So totally $(4!)^2$ ways for case. Similarly, we have the same number for case 2. So we have totally $2(4!)^2$ ways.

(iv) there are 4 married couples and each couple must sit together.

Solution. Use the similar idea from (ii) to get $2^44!$ ways.

3. Suppose that each of n sticks is broken into one long and one short part. The $2n$ parts are then shuffled and arranged into n pairs from which new sticks are formed (assume that the shuffling is such that all possible n pairs are equally likely). Find the probability that (justifying your answers in all three cases)

(i) the parts will be joined into their original form;

Solution. The total number of ways to get n pairs is $\frac{(2n)!}{2^n n!}$. There is only one way to pair them up into their original form. Then the probability is $\frac{1}{\frac{(2n)!}{2^n n!}}$.  

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(ii) all long parts are paired with short parts;

**Solution.** The total number of ways to pair them up (a long part and a short part are paired) is $n!$. Then the probability is $\frac{n!}{2^n n!}$.

(iii) at least one of the original sticks is formed.

**Solution.** Consider the probability that no original sticks are formed, denoted by $P(n)$. We shall condition on how the long part of the first stick is paired.

If the long part of the first stick is paired with a long part (the probability is $\frac{n-1}{2n-1}$), then conditioning on this event, the conditional probability is $P(n - 1) + \frac{1}{2n-3} P(n - 2)$. The first term is for the case the two short parts to which their long parts are paired up are not paired up. Then we can treat one short part among two as a long part to have $n - 1$ sticks case. The second term is for the case the two short parts to which their long parts are paired up are also paired up. Then remaining parts are from $n - 2$ sticks.

If the long part of the first stick is paired with a short part which is not the short part of the first stick (the probability is $\frac{n-1}{2n-1}$), then conditioning on this event, the conditional probability is also $P(n - 1) + \frac{1}{2n-3} P(n - 2)$ by a similar reasoning.

Then we have $P(n) = \frac{2n-2}{2n-1} (P(n - 1) + \frac{1}{2n-3} P(n - 2))$. Use this recursive formula to solve $P(n)$.

4. The solution to the Birthday Problem discussed in class shows us that if there are 23 or more people in a room, then the probability that at least two of them have the same birthday exceeds 1/2. On learning of this, a certain student, Peter, believes that with probability greater than 1/2 someone in the next 22 people that he meets will have the same birthday as him. State whether or not you agree with Peter. If yes, state why and if not, explain the fallacy in Peters thinking and calculate the minimum number of people that Peter must meet before the probability is greater than 1/2 that someone will share his birthday.

**Solution.** We need to look at the probability that Peter need to meet $i$ people to have one person (the last person) to share the same birthday with him. that probability is $\left(\frac{364}{365}\right)^{i-1} \frac{1}{365}$. We need to solve the minimum $n$ such that $\sum_{i=1}^{n} \left(\frac{364}{365}\right)^{i-1} \frac{1}{365} \geq 1/2$.

5. If $A$, $B$ and $C$ are independent events, show that $A$ is independent of $B \cup C$ and also that $A \setminus B$ is independent of $C$. Justify each step of your proofs.
Solution. Prove $P(A \cap (B \cup C)) = P(A)P(B \cup C)$ first.

$$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$$
$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$
$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$$
$$= P(A)(P(B) + P(C) - P(B \cap C))$$
$$= P(A)P(B \cup C).$$

Note $A \setminus B = A \cap B^c$. Prove $P((A \cap B^c) \cap C) = P(A \cap B^c)P(C)$. The detail is omitted.

6. A regular tetrahedron, three faces of which are painted red, yellow and blue, respectively, and
the fourth with all the three colors, is thrown onto the plane in a fashion such that it is equally
likely to fall on any of its four faces. Denote $R$, $Y$ and $B$, respectively to be the event that the
tetrahedron lands so that the red, yellow or blue color, respectively, touches the plane. Are the
events $R$, $Y$ and $B$ pairwise independent? Are the events $R$, $Y$ and $B$ independent?

Solution. Note that there are two possibilities for a color touches the plane. Hence $P(R) =
P(Y) = P(B) = 1/2$. Only when face 4 touches the plane, we have more than one color touches
the plane. Hence $P(RY) = P(RB) = P(YB) = P(RYB) = 1/4$. It is easy to see that $R$, $Y$ and
$B$ are pairwise independent, but $R$, $Y$ and $B$ are not independent since $P(RYB) = 1/4 \neq 1/8 =
P(R)P(Y)P(B)$.

7. A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability
that the hand contains at least one card from each of the four suits?

Solution. There are totally $\binom{52}{5}$ ways to get a 5-card hand. To get a hand with at least one
card from each of the four suits, we first decide which suit we shall select two cards. There are
4 ways to select the suit. Then for the rest 3 suits, we just select one card. So totally we have
$4 \times \binom{13}{2} \times 13^3$ ways. Then the probability is $4 \times \binom{13}{2} \times 13^3 / \binom{52}{5} = 0.2637$.

8. Five balls are randomly chosen, without replacement, from an urn that contains 5 red, 6 white,
and 7 blue balls. Find the probability that at least one ball of each color is chosen?

Solution. Let $R, W, B$ denote, respectively, the events that there are no red, no white, and no
blue balls chosen. Then

$$P(R \cup W \cup B) = P(R) + P(W) + P(B) - P(RW) - P(RB) - P(WB) + P(RWB)$$
\[
\begin{align*}
&= \binom{13}{5} \binom{12}{5} \binom{11}{5} \binom{7}{5} \binom{6}{5} - \binom{18}{5} \binom{18}{5} \binom{18}{5} \binom{18}{5} \binom{18}{5} \\
&= 0.2933.
\end{align*}
\]

Then the probability that all colors appear in the chosen balls is approximately \(1 - 0.2933 = 0.7067\).

9. If \(N\) people, including \(A\) and \(B\), are randomly arranged in a line, what is the probability that \(A\) and \(B\) are next to each other? What would the probability be if the people were randomly arranged in a circle?

**Solution.** The total number of ways to arrange \(N\) people in a line is \(N!\). The first part is similar to 2(ii). The number of ways is \(2(N - 1)!\). Hence the probability is \(\frac{2(N-1)!}{N!}\). For the second part, The total number of ways to arrange \(N\) people in a circle is \((N - 1)!\). The number of ways to have \(A\) and \(B\) being together is \(2(N - 2)!\). Then the probability is \(\frac{2(N-2)!}{(N-1)!}\).