Show FULL JUSTIFICATION for all your answers.

1. A batch of one hundred items is inspected by testing four randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted if it contains five defectives?

**Solution.** Let $A$ be the event that the batch is accepted and $B$ be the event that the batch contains five defectives. We are looking for $P(A|B)$. In fact,

$$P(A|B) = \frac{\binom{95}{4}}{\binom{100}{4}}.$$

2. Two out of three prisoners are to be released. One of the prisoners asks a guard to tell him the identity of a prisoner other than himself that will be released. The guard refuses with the following rationale: at your present state of knowledge, your probability of being released is $2/3$, but after you know my answer, your probability of being released will become $1/2$, since there will be two prisoners (including yourself) whose fate is unknown and exactly one of the two will be released. What is wrong with the guard’s reasoning?

**Solution.** Let $A$ and $B$ are the two prisoners other than “himself”. Suppose that the guard says $A$ will be released, denoted by $A$. Let $a$ (resp. $b$, he) stands for the event that $A$ (resp. $B$, he) will not be released. By Bayes’ Rule,

$$P(\text{he}|A) = \frac{P(\text{he and } A)}{P(A)} = \frac{P(A|\text{he})P(\text{he})}{P(A|\text{he})P(\text{he}) + P(A|a)P(a) + P(A|b)P(b)} = \frac{1/2 \times 1/3}{1/2 \times 1/3 + 0 \times 1/3 + 1 \times 1/3} = \frac{1}{3}.$$

Then $P(\text{he will be released}|A) = 2/3$. The fate of “he” is not changed. The problem is due to the extra randomness from the inquiry by one of the prisoners.
3. A bin contains 3 different types of disposable flashlights. The probability that a type 1 flashlight will give over 100 hours of use is 0.7, with the corresponding probabilities for type 2 and type 3 flashlights being 0.4 and 0.3, respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

(a) What is the probability that a randomly chosen flashlight will give more than 100 hours of use?

Solution. Let $A$ be the event that a randomly chosen flashlight will give more than 100 hours of use.

$$P(A) = P(A|\text{type 1})P(\text{type 1}) + P(A|\text{type 2})P(\text{type 2}) + P(A|\text{type 3})P(\text{type 3}) = 0.7 \cdot 0.2 + 0.4 \cdot 0.3 + 0.3 \cdot 0.5 = 0.41.$$ 

(b) Given the flashlight lasted over 100 hours, what is the conditional probability that it was a type 1 flashlight?

Solution. By Bayes’ rule,

$$P(\text{type 1}|A) = \frac{P(\text{type 1})P(A|\text{type 1})}{P(A|\text{type 1})P(\text{type 1}) + P(A|\text{type 2})P(\text{type 2}) + P(A|\text{type 3})P(\text{type 3})} = \frac{0.2 \cdot 0.7}{0.41} = \frac{14}{41}.$$ 

4. Suppose that we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

Solution. Let $A$ be the event that the upper side the card is red. We need to compute $P(\text{bottom side is black}|A)$

$$P(\text{bottom side is black}|\text{upper side is red}) = \frac{P(\text{bottom side is black and upper side is red})}{P(A)} = \frac{\frac{1}{2}P(\text{card 3})}{P(A|\text{card 1})P(\text{card 1}) + P(A|\text{card 2})P(\text{card 2}) + P(A|\text{card 3})P(\text{card 3})} = \frac{\frac{1}{6}}{1/3 + 0 + 1/6} = \frac{1}{3}. $$
5. An urn contains 7 balls, of which one is special. If 4 of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at that time, what is the probability that the special ball is chosen?

**Solution.** Let $A$ be the event that the special ball is chosen. We first consider $P(A^c)$. $A^c$ means that the special ball will not be chosen among the first 4 ball selected. then

$$P(A^c) = \frac{6 \cdot 5 \cdot 4 \cdot 3}{7 \cdot 6 \cdot 5 \cdot 4} = \frac{3}{7}.$$ 

Hence $P(A) = 1 - P(A^c) = \frac{4}{7}$.

6. If A flips $n+1$ and B flips $n$ fair coins, what is the probability that A gets more heads than B?

**Solution.** Apply the total probability law by considering the result from the first coin flip from A. then

$$P(A \text{ gets more heads}) = P(A \text{ gets more heads}|H)P(H) + P(A \text{ gets more heads}|T)P(T)$$

$$= \frac{1}{2}(P(A \text{ gets more heads}|H) + P(A \text{ gets more heads}|T)).$$

Once A gets a head in the first flip, then $P(A \text{ gets more heads}|H)$ is the same as the probability that A gets at least the same number of heads as B if A and B both flip $n$ coins, and it is the same as 1– the probability that B gets more heads than A if A and B both flip $n$ coins. Since the probability that B gets more heads than A if A and B both flip $n$ coins is the same as the probability that A gets more heads than B if A and B both flip $n$ coins, then $P(A \text{ gets more heads}|H) + P(A \text{ gets more heads}|T) = 1$. Hence $P(A \text{ gets more heads}) = \frac{1}{2}$.

7. Show that if $P(A|B) = 1$, then $P(B^c|A^c) = 1$.

**Solution.** $P(B^c|A^c) = \frac{P(B^c \cap A^c)}{P(A^c)} = \frac{P((B \cup A)^c)}{P(A^c)} = \frac{1 - P(B \cup A)}{1 - P(A)}$. Since $1 = P(A|B) = \frac{P(A \cap B)}{P(B)}$, then $P(B) = P(A \cap B)$. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, then $P(A \cup B) = P(A)$. Hence $1 - P(A \cup B) = 1 - P(A)$, then we have the desired result.

8. Two gamblers, A and B, bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, A collects 1 dollar from B, whereas if it comes up tails, A pays 1 dollar to B. They continue to do this until one of them runs out of money. If it is assumed each flip results in a head with probability 0.6, what is the probability that A ends up with all the money if he starts with 10 dollars and B also starts with 10 dollars.

**Solution.** See Problem 37 for the solution.