Stochastic Processes
Sigma-Fields
Stopping Times

Setup
\((\Omega, \Sigma, P)\) : probability space.

\(\Omega\) : sample space
\(\Sigma\) : collection of events (sigma-algebra)
\(P\) : probability function \(P : \Sigma \rightarrow [0,1]\) (countably additive)

Stochastic Process
Collection of random variables indexed by time
\(X : T \rightarrow S\)

\(T\) : index set \((T = \{0, 1, 2, \ldots\}\) "discrete"
or \(T = (0, \infty)\) "continuous time"

\(S\) : state space.
S: Polish space (complete, separable metric space w/ Borel \( \sigma \)-algebra \( \mathcal{B}(S) \))

\[ S = \mathbb{R} \times \mathbb{R} \times \mathcal{C}[0,1] \times \text{etc.} \]

For the time being, assume \( S = \mathbb{R} \).

\( X: \text{s.t. } X_t \text{ is a r.v. } \forall t \in \mathbb{T} \).

\( X_t : \mathbb{T} \rightarrow \mathbb{R} \text{ s.t. } X_t \text{ is } \mathcal{F}_t / \mathcal{B}(\mathbb{R}) \text{ mbl.} \)

\(\{ X_t \in A | Z \in \mathcal{F} \land A \in \mathcal{B}(\mathbb{R}) \} \).

For fixed \( w \in \mathcal{W} \), the map \( x \mapsto X_t(w) \) from \( \mathbb{T} \) to \( \mathbb{R} \) is the sample path of \( X(w) \).
3) For continuous time processes, we say that $X$ is (left, right) continuous if the sample paths are (left, right) continuous with probability one.

Let $X, Y$ be two stochastic processes.

Q: What do we mean by $X = Y$?

0) $X_t(w) = Y_t(w)$ $\forall$ $t \leq 0$, we say too strong, $X, Y$ truly are the same.

1) Indistinguishable

$$P(X_t = Y_t \land t \leq 0) = 1$$

2) Modification

$$P(X_t = Y_t) = 1 \land t \geq 0$$

3) Some finite-dimensional distributions (fields)

$$P((X_{t_1}, \ldots, X_{t_n}) \in A) = P((Y_{t_1}, \ldots, Y_{t_n}) \in A) = 1$$
for \( x_1, \ldots, x_n \geq 0 \), \( A \in B(\mathbb{R}^n) \), \( n = 1, 2, 3, \ldots \)

Clearly

1) \( \Rightarrow \) 2) \( \Rightarrow \) 3)

But converses not true:

1) 2) \( \Rightarrow \) 1) \( X_x = 0 \), \( Y_x = 1 \) if \( Z \sim \text{Exp}(1) \)
   
   \[ P(Y_x = 0) = 1 - P(Z = x) = 0. \]

2) 3) \( \Rightarrow \) 2) \( X_x = \begin{cases} Z & x = 1 \\ 0 & x \neq 1 \end{cases} \)
   \[ Y_x = \begin{cases} -Z & x = 1 \\ 0 & x \neq 1 \end{cases} \]
   
   for \( Z \sim N(0, 1) \).

Also, note that \( X, Y \) can be defined on different prob. spaces and still have the same tails.
Good News: with some path regularity,

$$2) \Rightarrow 1)$$

If $\{x_i: i \text{ continuous path}\}$ follows from

$$\{ X_t = Y_t \land t \in \Omega \} = \bigcup_{t \in \Omega \land \alpha} \{ X_t = Y_t \},$$

- left, right cont. OK too.

Filtrations:

We think of a stochastic process as evolving through time.

We also heuristically think of "information" evolving through time as well.

- how to make this formal?

A collection of $\sigma$-algebras $\{F_t\}_{t \in \mathbb{T}}$ is a filtration if
1) $\mathcal{F}_t \subseteq \mathcal{F}_{\infty}$ is a $\sigma$-alg. $A \in \mathcal{F}_t$

2) $\mathcal{F}_t \subseteq \mathcal{F}_{\infty}$ if $t \leq \Delta$.

$\mathcal{F} = \{0, 1, 2, \ldots\}$ "discrete filtration"

$\mathcal{F} = [0, \omega)$ "continuous filtration"

Work primarily w/ continuous filtrations.

Example:

$X :$ cont. time stochastic process

For each $x \in \mathcal{X}$ set $\mathcal{F}_t^x = \sigma(X_s : s \leq t)$

then $\{\mathcal{F}_t^x : t \geq 0\}$ is the filtration generated by $X$.

$A \in \mathcal{F}_t^x \implies \text{we know if } A \text{ has occurred by } t.$
Write \( F = \mathcal{F}_x \mathcal{F}_x^0 \) for the filtration \( (\mathcal{F}_x \text{ for natural filtration}) \)

As with processes, there is a notion of (left, right) continuity for filtrations.

1) \( \mathcal{F}_x^- = \sigma(\bigcup_{a < x} \mathcal{F}_a) \) \( \mathcal{F}_0^- = \mathcal{F}_0 \)
   - collection of events immediately prior to \( x \).

2) \( \mathcal{F}_x^+ = \bigwedge_{a > 0} \mathcal{F}_{x+a} \)
   - collection of events immediately after \( x \).

Say \( F \) is
1) right continuous if \( \mathcal{F}_x = \mathcal{F}_x^+ \)
2) left continuous if \( \mathcal{F}_x = \mathcal{F}_x^- \)
3) continuous if \( Z_t = Z_{t+} = Z_t^- \)

\[\begin{align*}
X_t &= 0 & t < 1 & Z \sim N(0, 1) \\
X_t &= 2 & t \geq 1
\end{align*}\]

\[\Rightarrow \exists X_t = Z_{t+} \quad Z_1^+ \neq Z_1^-\]

**Measurability and Adaptivity.**

Let \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) be a given filtered probability space.

Let \(X = \{X_t\}_{t \geq 0}\) be a stochastic process.

\(X_t : \Omega \to \mathbb{R}\) is \(\mathcal{F}_t / \mathcal{B}(\mathbb{R})\) mbl.

We now develop notions of measurability for the process \(X\) taking into account the filtration.
First, we say that $X$ is mbl if 

$$X(t, \omega) : \mathcal{D}_\omega \times \Omega \rightarrow \mathbb{R}$$

is 

$$\mathcal{B}(\mathcal{D}_\omega \times \Omega) / \mathcal{B}(\mathbb{R}) \text{ mbl.}$$

Joint map is mbl, not just each $X_t$.

Fubini: $\int A dX_t(\omega)$ is mbl $X \mapsto E[X_t]$ is mbl provided $E[|X_t|] < \infty \ \forall t$.

Next, we connect $X$ with the filtration $\mathcal{F}_t$.

We say $X$ is adapted to $\mathcal{F}_t$ if

$X_t$ is $\mathcal{F}_t$ mbl $\forall t$.

$$\{X_t, t \in \mathcal{T} \text{ } \mathcal{F}_t \} \text{ } \forall \mathcal{T} \in \mathcal{B}(\mathbb{R})$$

information in $\mathcal{F}_t$ sufficient to know $X_t$. 
We say $X$ is progressively measurable if

$$X(\omega) : \mathcal{F}_\infty \times \mathcal{F}_\infty \mapsto \mathcal{F}$$

is $\mathcal{F}_\infty \times \mathcal{F}_\infty \mapsto \mathcal{F}$. $\mathcal{F}_\infty \times \mathcal{F}_\infty \mapsto \mathcal{F}$

$$\{ (\omega) \mid \exists \xi \in \mathcal{A}, X_\xi(\omega) \in \mathcal{A} \} \in \mathcal{B}(\mathcal{F}_\infty \times \mathcal{F}_\infty)$$

Now, if $X$ is adapted to $\mathfrak{F}$ then

$$\forall \xi \in \mathfrak{F} : \{ X_\xi \in \mathcal{A} \} \in \mathcal{B}(\mathfrak{F})$$

but progressive measurability allows us to consider all the $\xi \in \mathfrak{F}$.

Similarly to the relationship between indistinguishability and modification in the presence of path-regularity, adaptivity implies progressive measurability.
L. If $X$ is (left, right) continuous and adapted, it is progressively mbl.

P. (right-continuity).

Sat. \[ X_n^+(w) = X \frac{(k+1)x(t)}{\delta^n} \Delta e \left( \frac{kt}{\delta^n} \right) \]

$k = 0, \ldots, m-1 \quad n = 1, 2, 3, \ldots$

$X_n^+(w) = X_n(w)$.

Since

\[ \frac{1}{\delta^n} e \left( \frac{kt}{\delta^n} \right) \in B \left[ 0, 1 \right] \]

\[ \frac{1}{\delta^n} X_n^+(w) \in A \subseteq \frac{1}{\delta^n} \frac{(k+1)x(t)}{\delta^n} \subseteq \frac{1}{\delta^n} x(t) \]

$X^+$ is progressively mbl.

But \[ X_n^+(w) \to X_+^+(w) \quad A \quad w. \]

\[ \therefore X \text{ is prog. mbl.} \]
Message: paths of $X$ have to be really wild for adaptivity to not imply progressability.

Stopping Times.

We would like to evaluate the process, not only at fixed times, but also at "random times" corresponding to events of interest.

E.g., whereas is $X$ the first time $X$ falls out of some set.

How can we do this?
A random time \( T \) is a map
\[ T: \mathcal{F} \rightarrow [0, \infty] \]
which is \( \mathbb{F} \)-measurable
- just a non-negative r.v. which may be infinite.

If \( T \) a random time and \( X \) a continuous time stochastic process (jointly measurable in \( t, \omega \)) define
\[ X_T(\omega) \equiv X_T(\omega)(\omega) \text{ on } \{ T < \infty \} \]

\[ \Rightarrow X_T \text{ a r.v. since compositions of r.v. are r.v.} \]

Now, consider \( T \) and \( \mathcal{F} \) (continuous time) filtration \( \mathcal{F} \). We would like to know for each \( t \), if \( T \) has occurred by \( t \), using only the accumulated information.
"Stopping Times" are precisely those random times for which we can do this.

We say a random time $T$ is a stopping time for $\mathcal{F}$ if

$$\{T \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0$$

We say $T$ is an optional time for $\mathcal{F}$ if

$$\{T < t\} \in \mathcal{F}_t \quad \forall t > 0$$

**Proposition.**

a) Stopping $\Rightarrow$ Optional

b) Stopping $\iff$ Optional if $\mathcal{F}$ is $\mathcal{T}$-continuous.
\[ \begin{align*} 
&\quad (i) \quad \exists T < x_3 = \bigcup_{n} \{ T \leq x - \frac{1}{n} \} \\
\quad &\quad (ii) \quad \exists T \leq x_3 = \bigvee_{m} \{ T < x + \frac{1}{m} \} \quad \forall m > 0 \}
\end{align*} \]

Basic Properties of Optimal/Stopping Times.

a) \( T \) optimal, \( \Theta > 0 \) constant
   \[ \Rightarrow T + \Theta \text{ stopping} \]
   - easy

b) \( T \) stopping
   \[ \Rightarrow T + \Theta, TVS, T + S \text{ stopping} \]

**Note**
\[ \{ T + S > x \} = \{ T = 0, S > x \} \\
\cup \{ S = 0, T > x \} \cup \{ T > 0, S > 0 \} \\
\cup \{ 0 < T < x, S + T > x \}. \quad (\text{check!}) \]

= \{ T = 0, S > x \} \cup \{ S = 0, T > x \} \\
\cup \{ T > 0, S > 0 \} \cup \left( \bigcup_{n \in \mathbb{Q}} \{ n < T < x, S > x - n \} \right) \\
\cup \left( \bigcup_{0 < n < x} \{ n < T < x, S > x - n \} \right) \\
\quad (\text{check!}) \quad \Box .

(1) \text{ Let } \{ T_n \}_{n=1}^{\infty} \text{ be optimal. Then } \\
\sup_n T_n \geq \inf_n T_n \text{ and } \lim_{n \to \infty} T_n \text{ are all optimal.}

(2) \[ \{ \sup_n T_n \geq x \} = \bigcap_{n} \{ T_n \geq x \} \]

(3) \[ \forall n > 0 \quad \{ \lim_{n \to \infty} T_n \geq x \} = \bigcap_{n} \left( \bigcup_{m=1}^{n} \{ \sup_{n \geq m} T_n \leq x + 1/n \} \right) \]
- Similar identities for \( \inf T_n \) vs \( \lim T_n \).

Q. If the \( T_n \) are stopping, which of the above are as well?

Important Stopping Times.

\( X \): right stochastic process taking values in \( \mathbb{R}^d \).

\( \mathcal{F} = F^X \)

\( \mathcal{P} \): Borel subset of \( \mathbb{R}^d \)

Set

\[ \tau_{\mathcal{P}} = \inf \{ t > 0 \mid X_t \in \mathcal{P} \} \]

- "hitting time" of \( X \) to \( \mathcal{P} \)
- we wish for this to be stopping
- but if we knew \( X \) then morally we should not know if \( X \) hit \( \mathcal{P} \).
- but we need some path regularity and set regularity.

**Proposition.**

a) \( \Gamma \) open, \( X \) \( \Gamma \)-continuous

\[ \Rightarrow \gamma_\Gamma \text{ optimal} \]

b) \( \Gamma \) closed, \( X \) continuous

\[ \Rightarrow \gamma_\Gamma \text{ stopping} \]

**Proof.**

a) \( \exists \gamma_\Gamma < x \) \( \Rightarrow \bigcup_{\delta \in \Delta \gamma} \{ x_\delta \in \Gamma \} \)

- show this.

b) Set \( \Gamma^n = \{ x \mid \inf_{\gamma \in \Gamma} |x - y| < \gamma \} \)

use a) + continuity.