EXCELLENT ABSTRACT ELEMENTARY CLASSES ARE TAME

RAMI GROSSBERG AND ALEXEI S. KOLESNIKOV

Abstract. We prove the statement from the title. As an application we conclude (using a theorem of Shelah):

Corollary 0.1. Suppose $V = L$. Let $T$ be a countable $L_{\omega_1,\omega}$ theory in a countable language. If $I(\aleph_{n+1}, T) < 2^{\aleph_n}$ for every $n < \omega$ then $K := \text{Mod}(T)$ is $\aleph_0$-tame (i.e. for any $p$ and $q$ distinct Galois types there exist a countable $M \in K$ such that $p \upharpoonright M \neq q \upharpoonright M$).

Introduction

In 1977 Shelah influenced by earlier work of Jónsson ([Jo1] and [Jo2]) in [Sh 88] introduced a semantic generalization of Keisler’s [Ke] treatment of $L_{\omega_1,\omega}(Q)$. It is the notion of Abstract Elementary Class:

Definition 0.2. Let $\mathcal{K}$ be a class of structures all in the same similarity type $L(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on $\mathcal{K}$. The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an abstract elementary class, AEC for short iff

A0 (Closure under isomorphism)
(a) For every $M \in \mathcal{K}$ and every $L(\mathcal{K})$-structure $N$ if $M \cong N$ then $N \in \mathcal{K}$.
(b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_1 : N_1 \cong M_1$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.

A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.

A2 Let $M, N, M^*$ be $L(\mathcal{K})$-structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.

A3 (Downward Löwenheim-Skolem) There exists a cardinal $\text{LS}(\mathcal{K}) \geq \aleph_0 + |L(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \geq A$ and $||N|| \leq |A| + \text{LS}(\mathcal{K})$.

A4 (Tarski-Vaught Chain)

Date: This very preliminary draft is from December 1, 2003.
1991 Mathematics Subject Classification. Primary: 03C45, 03C52. Secondary: 03C05, 03C95.

The research is part of the author’s work towards his Ph.D. degree under direction of Prof. Rami Grossberg. I am deeply grateful to him for his guidance and support.
(a) For every regular cardinal $\mu$ and every $N \in \mathcal{K}$ if $\{M_i : i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing (i.e. $i < j \implies M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.

(b) For every regular $\mu$, if $\{M_i : i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.

For $M$ and $N \in \mathcal{K}$ a monomorphism $f : M \rightarrow N$ is called an $\mathcal{K}$-embedding iff $f[M] \prec_{\mathcal{K}} N$. Thus, $M \prec_{\mathcal{K}} N$ is equivalent to “id$_M$ is a $\mathcal{K}$-embedding from $M$ into $N$”.

Many of the fundamental facts on AECs were introduced in [Sh 88], [Sh 394] and [Sh 576]. For a survey of some of the basics see [Gr1] or Chapter 13 of [Gr2].

In the late seventies Shelah proposed the following as a test problem:

**Conjecture 0.3** (Shelah’s conjecture). Let $\psi \in L_{\omega_1,\omega}$ be a sentence in a countable language. If $\psi$ is $\lambda$-categorical in some $\lambda \geq \beth_{\omega_1}$ then $\psi$ is $\mu$-categorical for every $\mu \geq \beth_{\omega_1}$.

In 1990 Shelah proposed a generalization for AECs:

**Conjecture 0.4** (see [Sh c]). Let $\mathcal{K}$ be an AEC. If $\mathcal{K}$ is categorical in some $\lambda \geq \text{Hanf}(\mathcal{K})$ then $\mathcal{K}$ is $\mu$-categorical for every $\mu \geq \text{Hanf}(\mathcal{K})$.

**Notation 0.5.** Let $\mu$ be a cardinal number and $\mathcal{K}$ a class of models. By $\mathcal{K}_\mu$ we denote the subclass $\{M \in \mathcal{K} : \|M\| = \mu\}$.

Two classical concepts that introduced in the fifties and studied extensively by Fraïssé, Robinson and Jonsson play also an important role in AECs:

**Definition 0.6.** Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an AEC and suppose $\mu \geq \text{LS}(\mathcal{K})$. We say that $\mathcal{K}$ has the $\mu$-amalgamation property iff for all $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 0, 1, 2$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) there exists $N^* \in \mathcal{K}_\mu$ and $f_\ell : M_\ell \rightarrow N^*$ (for $\ell = 1, 2$) such that $f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0$, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
M_1 & \xrightarrow{f_1} & N^* \\
\id & \downarrow & \uparrow f_2 \\
M_0 & \xrightarrow{\id} & M_2
\end{array}
\]

$M_0$ as above is called amalgamation base.

$\mathcal{K}$ has the $\mu$-joint mapping property iff for any $M_\ell \in \mathcal{K}_\mu$ for $\ell = 1, 2$ there are $N^* \in \mathcal{K}_\mu$ and $\mathcal{K}$-embeddings $f_\ell : M_\ell \rightarrow N^*$.

We say that $\mathcal{K}$ has the amalgamation property iff it has the $\mu$-amalgamation property for all $\mu \geq \text{LS}(\mathcal{K})$.

Using Axiom A0 from the definition of AEC it follows that both a stronger-looking and a weaker-looking amalgamation properties are equivalent to what we call above the amalgamation property:
Lemma 0.7. Let $K$ be an AEC. The following are equivalent

1. $K$ has the $\mu$-amalgamation property.
2. For all $M_\ell \in K_\mu$ (for $\ell = 0, 1, 2$) such that $M_0 \preceq_K M_\ell$ (for $\ell = 1, 2$) there exists $N^* \in K_\mu$ such that $N^* \succeq_K N_2$ and there is $f : M_1 \to N$ satisfying $f \upharpoonright M_0 = \text{id}_{M_0}$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{f} & N^* \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
M_0 & \xrightarrow{\text{id}} & M_2 \\
\end{array}
$$

3. For all $M_\ell \in K_\mu$ (for $\ell = 0, 1, 2$) such that $g_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) are $K$-embeddings there are $N^* \in K_\mu$ and there is $f_\ell : M_\ell \to N^*$ satisfying $f_1 \circ g_1 \upharpoonright M_0 = f_2 \circ g_2 \upharpoonright M_0$ i.e. the next diagram commutes:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{f_1} & N^* \\
\downarrow{g_1} & & \downarrow{f_2} \\
M_0 & \xrightarrow{g_2} & M_2 \\
\end{array}
$$

There are classical theorems of Robinson stating that if $T$ is a complete first-order theory than $\text{Mod}(T)$ has both the amalgamation and the joint mapping properties.

Galois types. In the theory of AECs the notion of complete first-order type is replaced by that of a Galois type:

**Definition 0.8.** Let $\beta > 0$ be an ordinal. For triples $(\bar{a}_l, M, N_l)$ where $\bar{a}_l \in \beta N_l$ and $M_l \preceq_K N_l \in K$ for $l = 0, 1$, we define a binary relation $E$ as follows: $(\bar{a}_0, M, N_0)E(\bar{a}_1, M, N_1)$ iff and there exists $N \in K$ and $K$-mappings $f_0, f_1$ such that $f_1 : N_1 \to N$ and $f_l \upharpoonright M_l = \text{id}_M$ for $l = 0, 1$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$:

$$
\begin{array}{ccc}
N_1 & \xrightarrow{f_1} & N \\
\downarrow{\text{id}} & & \downarrow{f_2} \\
M & \xrightarrow{\text{id}} & N_2 \\
\end{array}
$$

**Remark 0.9.** $E$ is an equivalence relation on the class of triples of the form $(\bar{a}, M, N)$ where $M \preceq_K N$, $\bar{a} \in N$ and both $M$ and $N$ are amalgamation bases. When $N$ is not an amalgamation base, $E$ may fail to be transitive, but the transitive closure of $E$ could be used instead.

**Definition 0.10.** Let $\beta$ be a positive ordinal.

1. For $M, N \in K$ and $\bar{a} \in \beta N$. The Galois type of $\bar{a}$ in $N$ over $M$, written $\text{ga-tp}(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$. 
2. We abbreviate \( \text{ga-tp}(\bar{a}/M, N) \) by \( \text{ga-tp}(\bar{a}/M) \).

3. For \( M \in \mathcal{K} \),
   \[
   \text{ga-S}^\beta(M) := \{ \text{ga-tp}(\bar{a}/M, N) \mid M \prec N \in \mathcal{K}_{||M||}, \bar{a} \in \beta N \}.
   \]
   We write \( \text{ga-S}(M) \) for \( \text{ga-S}^1(M) \).

4. Let \( p := \text{ga-tp}(\bar{a}/M', N) \) for \( M \prec \mathcal{K} M' \) we denote by \( p \restriction M \) the type \( \text{ga-tp}(\bar{a}/M, N) \). The domain of \( p \) is denoted by \( \text{dom}(p) \).

5. Let \( p = \text{ga-tp}(\bar{a}/M, N) \), suppose that \( M \prec \mathcal{K} N' \prec \mathcal{K} N \) and let \( \bar{b} \in \beta N' \) we say that \( \bar{b} \) realizes \( p \) iff \( \text{ga-tp}(\bar{b}/M, N') = p \restriction M \).

6. For types \( p \) and \( q \), we write \( p \leq q \) if \( \text{dom}(p) \subseteq \text{dom}(q) \) and there exists \( \bar{a} \) realizing \( p \) in some \( N \) extending \( \text{dom}(p) \) such that \( (\bar{a}, \text{dom}(p), N) = q \restriction \text{dom}(p) \).

In [GrV1] Grossberg and VanDieren introduced the notion of tameness as a candidate for a further "reasonable" assumption an an AEC that permits development of stability-like theory. In [GrV2] they recently proved the last step Shelah’s categorcity conjecture for tame AECs with the amalgamation property.

**Definition 0.11.** Let \( \mathcal{K} \) be an AEC with the amalgamation property and let \( \chi \geq \text{LS}(\mathcal{K}) \). The class \( \mathcal{K} \) is called \( \chi \)-tame iff

\[
\forall p, q \in \text{ga-S}(M) \quad (p \neq q \implies \exists N \prec \mathcal{K} M \text{ of cardinality } \leq \chi \text{ such that } p \restriction N \neq q \restriction N)
\]

for any \( M \in \mathcal{K}_{\geq \chi} \) and every \( p, q \in \text{ga-S}(M) \).

\( \mathcal{K} \) is tame iff it is \( \chi \)-tame for some \( \chi < \text{Hanf}(\mathcal{K}) \).

Suppose \( \mu > \chi \). The class is \((\chi, \mu)\)-tame iff

\[
\forall p, q \in \text{ga-S}(M) \quad (p \neq q \implies \exists N \prec \mathcal{K} M \text{ of cardinality } \leq \chi \text{ such that } p \restriction N \neq q \restriction N)
\]

for any \( M \in \mathcal{K}_\mu \) and every \( p, q \in \text{ga-S}(M) \).

In [Sh 394] Shelah proved that for an AEC with the amalgamation property. If \( \mathcal{K} \) is \( \lambda \)-categorical for some \( \lambda > \text{Hanf}(\mathcal{K}) \) then it is \((< \text{Hanf}(\mathcal{K}), \mu)\)-tame for all \( \text{Hanf}(\mathcal{K}) < \mu < \lambda \).

**Definition 0.12.** Let \( I \) be a subset of \( \mathcal{P}(n) \) for some \( n < \omega \) that is downward closed (i.e. \( t \in I \) and \( s \subseteq t \) implies \( s \in I \)).

For an \( \mathbf{S} = \langle M_s \mid s \in I \rangle \) is an \( I \)-system iff for all \( s, t \in I \)

1. \( s \leq t \implies M_s \prec \mathcal{K} M_t \) and
2. \( M_{s \cap t} = M_s \cap M_t \)

\( \mathbf{S} \) is a \((\lambda, I)\)-system iff in addition all the models are of cardinality \( \lambda \).

Denote by

\[
A^\mathbf{S}_t := \bigcup_{s \leq t} M_s
\]

Some sets that are amalgamation bases play in important role since they permit existence of Galois-types over them. Here is the formal
Definition 0.13. Let $\mathcal{K}$ be an AEC and suppose $\mu \geq \text{LS}(\mathcal{K})$. Suppose $S = \langle M_s \in \mathcal{K}_\mu \mid s \in I \rangle$ is an $I$-system for $I \subseteq \mathcal{P}(n)$ and $t \in \mathcal{P}(n)$. For $A := A^S_t$ we say that the set $A$ is an amalgamation base iff for all $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 0, 1, 2$) such that $A \subseteq |M_\ell|$ (for $\ell = 1, 2$) there exists $N^* \in \mathcal{K}_\mu$ such that $N^* \succ \mathcal{K} M_2$ and there is a $\mathcal{K}$-embedding $f : M_1 \to N$ satisfying $f \upharpoonright M_0 = \text{id}_A$, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
M_1 & \xrightarrow{f} & N^* \\
\downarrow\text{id}_A & & \downarrow\text{id}_{N^*} \\
A & \xrightarrow{\text{id}_A} & M_2
\end{array}
\]

By $\text{id}_A : A \to M_\ell$ we mean that $M_s \prec \mathcal{K} M_\ell$ holds for $\ell = 1, 2$ and every $s < t$.

Notation 0.14. Denote by $\text{Ab}(\mathcal{K})$ the class $\{A \mid A$ is an amalgamation base$\}$. Thus $\mathcal{K}$ has the $\lambda$-amalgamation property iff $\mathcal{K}_\lambda \subseteq \text{Ab}(\mathcal{K})$.

Clearly under the assumption that $\mathcal{K}$ has the amalgamation property the notion of Galois-type can be extended to include also $\text{ga-tp}(\bar{a}/A, M)$ for $A \in \text{Ab}(\mathcal{K})$.

Examples 0.15. (1) Let $T$ be a complete first-order theory and $\mathfrak{C}$ its monster model. By Robinson’s consistency lemma any $A^S_t$ for an $I$-system is an element of $\text{Ab}(\text{Mod}(T))$.

(2) On can prove that if $\mathcal{K}$ is the class of atomic models of a first-order $T$ satisfying all the assumptions of [Sh 87a] then $A \in \text{Ab}(\mathcal{K})$ iff $A$ is good.

1. The basic framework and concepts

Definition 1.1. A pair $\langle \mathcal{K}, \perp \rangle$ is a weak forking notion iff $\mathcal{K}$ is an AEC, where $\perp$ is a three-place relation called non-forking $\bar{a} \perp B$ for $\bar{a} \in \beta M$ for some $M \in \mathcal{K}$ and $A \subseteq B$ both elements of $\text{Ab}(\mathcal{K})$ such that $\perp$ is invariant under automorphisms which means for any $\bar{a}, A, B$ as above for all $N \in \mathcal{K}$ containing $A \cup B \cup \bar{a}$ we have that

$\bar{a} \perp B \iff f(\bar{a}) \perp f(B)$ \quad for all $f \in \text{Aut}(N)$.

the following conditions hold:

(0) Definability: There exists a cardinal number $\kappa$ such that the relation $\bar{a} \perp B$ is (set-theoretically) definable over $\kappa$ i.e. there is a f.o. formula $\varphi(x)$ in the similarity type $\text{LS}(\mathcal{K}) \cup \{\in, P, Q\}$ such that

$\langle H(\chi), \in, \kappa, A, B, \psi(y) \rangle_{\psi(y) \in \text{Fml}(\text{L}(\mathcal{K}))} = [\varphi[a] \iff a \perp B \quad for all finite a \in \bar{a}$.
(1) **Disjointness:**
\[
\bar{a} \perp B \implies \bar{a} \cap B \subseteq A.
\]

(2) **Existence:** Let \( A \in \text{Ab}(\mathcal{K}) \) if \( \bar{a} \) is such that there exists a model \( N \) containing \( B \) but disjoint to \( \bar{a} \) then \( \bar{a} \perp A \).

(3) **Extension property:** If \( \bar{a} \perp B \) then for all \( C \in \text{Ab}(\mathcal{K}) \) such that \( C \supseteq B \) there exists \( \bar{a}' \) in some \( M \in \mathcal{K} \) such that
\[
\bar{a}' \perp C \quad \text{and} \quad \text{ga-tp}(\bar{a}/A) = \text{ga-tp}(\bar{a}'/A).
\]

(4) **Symmetry:** if \( \bar{a} \perp A \bar{b} \), then \( \bar{b} \perp A \bar{a} \).

**Examples 1.2.**

(1) Let \( \mathcal{K} := \text{Mod}(T) \) when \( T \) is a first-order complete theory, \( \prec_{\mathcal{K}} \) is the usual elementary submodel relation and \( \perp \) is the non-forking relation.

Clearly \( \langle \mathcal{K}, \prec_{\mathcal{K}} \rangle \) is a weak forking notion iff \( T \) is simple. \( \kappa \) in this case is \( \kappa(T) \).

(2) Let \( \mathcal{K} := \text{Mod}(T) \) when \( T \) is a first-order complete theory, \( \prec_{\mathcal{K}} \) is the usual elementary submodel relation and \( \perp \) is the non-dividing relation. It is not difficult to show that \( \langle \mathcal{K}, \prec_{\mathcal{K}} \rangle \) is a weak forking notion with \( \kappa = \aleph_0 \) iff \( T \) is supersimple.

(3) Let \( T \) be a countable first-order theory, suppose that \( T \) is \( \aleph_0 \)-atomically stable, i.e. for \( R[p] < \infty \) for every atomic type, let
\[
\mathcal{K} := \{ M \models T \mid \text{ga-tp}(a/\emptyset, M) \text{ is an isolated type for every } a \in |M| \}.
\]

Where \( p \in S(A) \) is called atomic iff \( A \cup \{a\} \) is atomic subset of \( \mathcal{C} \) and \( a \models p \). An atomic type is stationary iff there is a finite \( B \subseteq A \) and a countable model \( N \) containing the set \( B \) and an atomic realization \( a \) of \( p \) we have that
\[
R[p] = R[\text{ga-tp}(a/B)] = R[\text{ga-tp}(a/N)].
\]

An atomic set \( A \subseteq \mathcal{C} \) is good iff for every consistent \( \varphi(x; a) \) (with \( a \in A \)) there is an isolated type \( p \in S(A) \) containing \( \varphi(x; a) \).

**Definition 1.3.** For \( M \in \mathcal{K}^a \) and \( a \in M \) define by induction of \( \alpha \) when \( R[\varphi(x; a)] \geq \alpha \)
\[
\alpha = 0; \quad M \models \exists x \varphi(x; a)
\]
For \( \alpha = \beta + 1; \)
There are \( b \supseteq a \) and \( \psi(x; b) \) such that
\[
R[\varphi(x; a) \land \psi(x; b)] \geq \beta
\]
\[
R[\varphi(x; a) \land \neg\psi(x; b)] \geq \beta \quad \text{and for every } c \supseteq a
\]
there is $\chi(x;c)$ complete s.t.

$$R[\varphi(x;a) \land \chi(x;c)] \geq \beta$$

**Notation 1.4.**

$$DA := \{\text{ga-tp}(a/A) \mid A \cup \{a\} \text{ is atomic}\}.$$ 

**Fact 1.5 ([Sh 87a]).** If $|DA| < 2^{\aleph_0}$ then $A$ is good.

(4) Let $\mathcal{K}$ be the class of elementary submodels of a sequentially homogeneous model. Let $M_1 \perp M_2$ stand for $\text{ga-tp}(a/M_2)$ does not strongly-split over $M_0$ for every $a \in |M_1|$.

Compare with XII.2 of [Sh c].

**Definition 1.6 (Stable systems).** Let $\langle \mathcal{K}, \perp \rangle$ be weak forking notion. Suppose $I \subseteq \mathcal{P}^-(n)$, suppose $S = \{M_s \mid s \in I\}$ is a $(\lambda, n)$-system. The system $S$ is called $(\lambda, n)$-stable iff for every enumeration $\bar{s} := (s(i) \mid i < m)$ of $I$ (always without repetitions such that $s(i_1) <_I s(i_2) \implies i_1 < i_2$)

(1) $A^S_{s(i)}$ is good for all $i$,

(2) for every $\langle b_i \in |M_s(i)| \mid i \leq j \leq m \rangle$ there are $\langle b'_i \in |M_s(i) \cap s(j)| \mid i \leq m \rangle$ such that

(a) \[\text{ga-tp}(b_0, b_1, \ldots /|M_0|) = \text{ga-tp}(b'_0, b'_1, \ldots /|M_0|)\]

and

(b) $s(i) \leq s(j) \implies b'_i = b_i$.

(3) $$A^S_{s(j)} \perp_{|M_s(j)| \mid i < j} \bigcup |M_s(i)|.$$ 

**Axiom 1.7 (Generalized Symmetry).** Let $\langle \mathcal{K}, \perp \rangle$ be weak forking notion. We say that $\langle \mathcal{K}, \perp \rangle$ has the $(\lambda, n)$-symmetry property iff for every $I \subseteq \mathcal{P}^-(n)$ and every $S = \{M_s \mid s \in I\}$ $(\lambda, n)$-system $S$. The system is $(\lambda, n)$-stable iff there exists an enumeration $\bar{s}$ of $I$ satisfying requirements (1), (2) and (3) of the previous definition.

CHECK if follows from symmetry.

**Definition 1.8 ($n$-dimensional amalgamation).** Let $\langle \mathcal{K}, \perp \rangle$ be weak forking notion, it has the $(\lambda, n)$-existence property iff for every stable system $S = \{M_s \mid s \in \mathcal{P}^-(n)\}$ of models of cardinality $\lambda$, there exists a model over the set $A^S_n$.

**Definition 1.9 (systems are amalgamation bases).** Let $\langle \mathcal{K}, \perp \rangle$ be weak forking notion, it has the $(\lambda, n)$-non-uniqueness property iff for every stable system $S = \{M_s \mid s \in \mathcal{P}^-(n)\}$ we have that $A^S_n \in \text{Ab}(\mathcal{K})$. 

Definition 1.10 (goodness). Let \( \langle K, \perp \rangle \) be weak forking notion, it has the \((\lambda, n)\)-goodness property iff \( \langle K, \perp \rangle \) has the \((\lambda, n)\)-symmetry property and for every stable system \( S = \langle M_s \mid s \in \mathcal{P}^-(n) \rangle \) of models of cardinality \( \lambda \), has the \((\lambda, n)\)-existence property and the \((\lambda, n)\)-non-uniqueness property.

Theorem 1.11 (characterizing goodness for f.o.). Let \( T \) be a complete countable f.o theory. Suppose \( T \) is superstable without dop. If \( S = \langle M_s \mid s \in \mathcal{P}^-(n) \rangle \) is a stable system of models of cardinality \( \aleph_0 \) then TFAE

1. the set \( A^n_S \) is an amalgamation base
2. There is a prime and minimal model over \( A^n_S \).

Definition 1.12 (excellence). Let \( \langle K, \perp \rangle \) be weak forking notion and let \( \lambda \geq \text{LS}(K) \). \( \langle K, \perp \rangle \) is \( \lambda \)-excellent iff \( \langle K, \perp \rangle \) has the \((\lambda, n)\)-goodness property for every \( n < \omega \). When \( \lambda = \text{LS}(K) \) we say that \( K \) excellent instead of \( \lambda \)-excellent.

Theorem 1.13 (Shelah 1982). Let \( T \) be a complete countable f.o theory. Suppose \( T \) is superstable without dop. TFAE

1. \( \langle \text{Mod}(T), \prec \rangle \) is excellent.
2. Mod(\( T \)) has the \((\aleph_0, 2)\)-goodness property.
3. \( T \) does not have the otop.

For proof see [Sh c]....

Remark 1.14. Even for complete first-order theories in general the \((\lambda, n)\)-amalgamation property may fail. Failure of \((\aleph_0, 3)\)-amalgamation is witnessed by the example of a triangle-free random graph. Start with a triple of models \( M_i, i < 3 \), and fix some elements \( a_i \in M_i \). Take a triple of models \( M_{01}, M_{02}, \) and \( M_{12} \) that form an \((\aleph_0, \mathcal{P}^-(3))\)-system over \( M_i \), and such that \( M_{ij} \models R(a_i, a_j) \) for \( i < j < 3 \). The system cannot be amalgamated since the amalgam would witness a triangle.

COMMENT: this example was suggested by Shelah. It is an example of a non-simple theory. It can be generalized to a failure of \((\aleph_0, n + 1)\)-amalgamation by using \( n \)-dimensional tetrahedron-free graphs. Those examples will be simple first order theories.

There is an example of a triple of totally categorical theories \( T_{ij}, i < j < 3 \), that are pairwise coherent, but cannot be “amalgamated” into a consistent first order theory, i.e., 3-dimensional Robinson’s consistency test fails.

For \( i < 3 \), let \( T_0 \) be the theory of an infinite set. For \( i < 3 \), let \( L_i := \{ P_i, f_i \} \), and \( T_i \) says that the model is divided by \( P_i \) into two parts of equal size, as witnessed by \( f_i \). For \( i < j < 3 \), \( T_{ij} \) contains the union of \( T_i \) and \( T_j \), and says that \( P_i(x) \iff \neg P_j(x) \). Then clearly the union \( \bigcup_{i<j<3} T_{ij} \) is inconsistent.

Fact 1.15 (Hart and Shelah 1986). For every \( n < \omega \) there is an \( \aleph_0 \)-atomically stable class \( K \) of atomic models of a countable f.o. theory such that \( K \) is has the \((\aleph_0, k)\)-goodness property for all \( k < n \) but is not excellent.
**Theorem 1.16.** If $\langle K, \sqsubseteq \rangle$ is excellent then it has the $(\lambda, n)$-goodness property for every $n < \omega$ and every $\lambda \geq \text{LS}(K)$.

*Proof.* Will be added. 

$\kappa$ is

2. *Tameness of AEC with $n$-amalgamation*

In this section, $\mathcal{K}$ is an AEC with 2-amalgamation and arbitrarily large models.

**Theorem 2.1.** If $\langle K, \sqsubseteq \rangle$ is excellent then $K$ is $\text{LS}(K)$-tame.

*Proof.* Let $\kappa$ be the least uncountable cardinal witnessing that $\mathcal{K}$ is not $\text{LS}(K)$-tame.

Thus there are $M, N_0, N_1$ be of size $\kappa$, $a_i \in N_i, i = 0, 1$, realize the same Galois type over every $K$-submodel of $M$ of size $\text{LS}(\mathcal{K})$ such that the Galois types of $a_0, a_1$ over $M$ are different. By renaming some of the elements we may assume that $N_0 \cap N_1 = M$.

By the existence and invariance properties we may assume that $N_0 \bumpeq M N_1$. Let $\chi$ be a regular cardinal large enough so that $N_1, N_0, M \in H(\chi)$ and also the definition of $\mathcal{K}$ is there as well as $\langle H(\chi), \in \rangle$ reflects all the relevant information e.g.

$$\langle H(\chi), \in \rangle \models N_0 \perp M N_1.$$ 

Now pick $\{\mathcal{B}_i \mathcal{C} H(\chi), M, N_0, N_1 \ldots \in \} | i < \kappa \}$ such that $\|\mathcal{B}_i\| = |i| + \text{LS}(\mathcal{K})$ and $\langle \mathcal{B}_j | j \leq i \rangle \in \mathcal{B}_{i+1}$ for all $i < \kappa$.

By minimality of $\kappa$, the Galois types of $a_0, a_1$ are the same over every $K$-submodel of $M$ of size less than $\kappa$. To get a contradiction, we construct a model $N_{01}$ and embeddings $f_0 : N_0 \rightarrow N_{01}$ and $f_1 : N_1 \rightarrow N_{01}$ that fix $M$ and map $a_0, a_1$ to the same sequence.

Let $\{\langle M^i, N^i_0, N^i_1 | i < \kappa \}$ be the interpretation of the corresponding models in $\mathcal{B}_i$, so we have

1. $a_\ell \in N^0_\ell$ for $\ell = 0, 1$;
2. $M^i, N^i_0,$ and $N^i_1$ are $\prec_\kappa$-increasing continuous chains of $\mathcal{K}$-models with union $M, N_0,$ and $N_1$ respectively.
3. $\|M^i\| = \|N^i_0\| = \|N^i_1\| = |i| + \text{LS}(\mathcal{K})$, for all $i < \kappa$.
4. $N^i_0 \perp N^i_1$ for all $i < \kappa, \ell = 0, 1$.

By induction on $i \leq \kappa$, define a model $N^i_{01}$ and embeddings $f^i_\ell : N^i_\ell \rightarrow N^i_{01}$, $\ell = 0, 1$. In addition, we need to keep track of embeddings $f^j_{01} : N^j_{01} \rightarrow N^{ij}_{01}$ such that $\{N^i_{01}, f^j_{01} | i < j < \alpha \}$ form a direct system of $\mathcal{K}$-submodels.
Base $i = 0$: since the Galois types of $a_0$, $a_1$ over $M^0$ coincide, there is a model $N^i_{0\alpha}$ and embeddings $f^i_\ell : N^i_\ell \to N^i_{0\alpha}$ that map $a_0$, $a_1$ together.

Successor step. We have $f^{i+1}_\ell : N^{i+1}_\ell \to N^{i+1}_{0\alpha}$ for $\ell = 0, 1$. We also have the identity embeddings $M^i \to M^{i+1}$, $N^i_\ell \to N^{i+1}_\ell$, $\ell = 0, 1$. The picture is:

Let $\lambda := |i| + \text{LS}(K)$. By 3-amalgamation, we get $N^{i+1}_{0\alpha}$ and embeddings $f^{i+1}_\ell : N^{i+1}_\ell \to N^{i+1}_{0\alpha}$ for $\ell = 0, 1$. For the direct system part, 3-amalgamation gives $N^i_{0\alpha} \prec K N^i_\alpha$ and $K$-embeddings $f^{i+1}_\ell : N^{i+1}_\ell \to N^{i+1}_{0\alpha}$.

Limit step. We have that $\{N^i_\alpha, f^i_{ij} \mid i < j < \alpha\}$ form an $\prec_K$-chain. Let $N^{\alpha}_{0\alpha}$ be the union and $f^\alpha_\ell$ be the union of the corresponding chain of $\prec_K$-embeddings. By Axiom A4 this is what we need.

Finally, the model $N^{\alpha}_{0\alpha}$, and the maps $f^\alpha_\ell$, $\ell = 0, 1$ are as needed. The image of $a_0$ under $f^\alpha_0$ is $f^\alpha_0(a_0) = f^\alpha_1(a_1)$, i.e., is the same as the image of $a_1$ under $f^\alpha_1$.

A similar proof gives several related theorems, e.g.:

**Theorem 2.2.** Let $\mathcal{K}$ be an AEC, and $\mu_0 > \text{LS}(\mathcal{K})$ if $\mathcal{K}$ has the $(\lambda, 3)$-AP for all $\text{LS}(\mathcal{K}) \leq \lambda < \mu_0$ then given $M \in \mathcal{K}_{\mu_0}$ for any $p \neq q \in \text{ga-S}(M)$ there is $N \prec_K M$ of cardinality $\text{LS}(\mathcal{K})$ such that $p \upharpoonright N \neq q \upharpoonright N$.

**Theorem 2.3.** Suppose that $\mathcal{K}$ has $(\aleph_0, n)$-amalgamation property for all $n < \omega$. Then $\mathcal{K}$ has $(\lambda, n)$-amalgamation for all $\lambda$.

**Proof.** The statement follows from the two claims:

**Claim 2.4.** Suppose that $\mathcal{K}$ has $(\lambda, n + 1)$-amalgamation. Then $\mathcal{K}$ has $(\lambda^+, n)$-amalgamation.
Corollary 2.6. Suppose that $\lambda$ is a limit cardinal and $\mathcal{K}$ has $(< \lambda, n+1)$-amalgamation. Then $\mathcal{K}$ has $(\lambda, n)$-amalgamation.

Indeed, $(\aleph_\alpha, n)$-amalgamation property for all $n < \omega$ for $\mathcal{K}$ implies $(\aleph_{\alpha+1}, n)$-amalgamation property for all $n < \omega$ for $\mathcal{K}$ by Claim 2.4. Claim 2.5 gives $(\aleph_\alpha, n)$-amalgamation property for limit $\alpha$, for all $n < \omega$.

Proof of Claim 2.4. Let $\{M_s \mid s \in \mathcal{P}^-(n)\} \subset \mathcal{K}_\lambda$ be an incomplete $n$-diagram of models in $\mathcal{K}$. Our goal is to find $M_n$ and the embeddings $\{f_s \mid s \subseteq n-1\}$, $f_s : M_s \to M_n$ that make the diagram commute.

Take $\{M^i_s \mid i < \lambda^+, s \in \mathcal{P}^-(n)\}$ a resolution of the incomplete $n$-diagram. We may assume that $|M^i_s| = \lambda$ for all $s$, $i$.

By induction on $i \leq \lambda^+$, define a model $M^i_n$ and embeddings $f^i_s : M^i_s \to M^i_n$, for each $s \subseteq n-1$. As before, we will keep track of embeddings $f^{ij}_n : M^i_n \to M^j_n$ such that $\{M^i_n, f^{ij}_n \mid i < j < \alpha\}$ form a direct system of $\mathcal{K}_\lambda$-submodels.

For the base case, we just take the completion of the $n$-diagram $\{M^0_s \mid s \in \mathcal{P}^-(n)\}$. It exists since we are assuming $(\lambda, n+1)$-amalgamation.

Successor step. We have $f^i_s : M^i_s \to M^i_n$ for $s \subseteq n-1$. We also have the identity embeddings $M^i_s \to M^{i+1}_s$, $s \subseteq n$. By $(\lambda, n+1)$-amalgamation, we get $M^{i+1}_n$ and embeddings $f^{ij+1}_n : M^{i+1}_n \to M^{j+1}_n$ for $s \subseteq n-1$. For the direct system part, $(\lambda, n+1)$-amalgamation also gives $f^{ij}_n : M^i_n \to M^{j+1}_n$. So we let $f^{ij+1}_n := f^i_n \circ f^{ij}_n$ for $j < i$, and $f^{ij+1}_n := f^i_n$.

Limit step. We have that $\{M^\alpha_n, f^{ij}_n \mid i < j < \alpha\}$ form a direct system. Let $M^\alpha_n$ be the direct limit of the system. As before, we define the maps from $M^\alpha_n$ to $M^\alpha_n$ by

$$f^\alpha_s := \bigcup_{i < \alpha} f^{i\alpha}_n \circ f^i_s$$

for $s \subseteq n-1$.

Finally, the model $M^\lambda_\alpha$, and the maps $f^\lambda_\alpha$, $s \subseteq n-1$, are as needed. \( \dashv \)

Proof of Claim 2.5. Is almost exactly the same, the only difference is that the cardinality of models in the resolution will be $|i| + \aleph_0$.

\( \dashv \)

From the two theorems above we easily get

**Corollary 2.6.** If $\mathcal{K}_{\aleph_0}$ has $n$-amalgamation property for all $n < \omega$, then $\mathcal{K}$ is $\aleph_0$-tame.

**References**


E-mail address, Rami Grossberg: rami@cmu.edu
URL: http://www.math.cmu.edu/~rami

Department of Mathematics, Carnegie Mellon University, Pittsburgh PA 15213

E-mail address: alexei@andrew.cmu.edu
URL: http://www.math.cmu.edu/~alexei

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213