EXAMPLES OF THE LAVREN'TIEV PHENOMENON
WITH CONTINUOUS SOBOLEV EXPONENT DEPENDENCE

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ABSTRACT. In this report, we demonstrate the existence of variational problems with infima that depend continuously upon the Sobolev space from which the competing functions are taken. It is shown, for each $\alpha$ in a particular class of continuous functions, that there is a variational integral and boundary conditions such that, for every $p \in [1, \infty]$, the infimum is equal to $\alpha(p)$ if the admissible class is a subset of $W^{1,p}$. So the manner in which the infimum depends upon the Sobolev exponent may be prescribed.

1. INTRODUCTION

In 1926, M. Lavrentiev [8] demonstrated, by example, that one could have a functional of the form $\int_a^b L(x, u(x), u'(x)) \, dx$ such that the infimum for the functional over the absolutely continuous functions is strictly less than its infimum over the continuously differentiable functions, when $u : \mathbb{R} \to \mathbb{R}$ is required to meet certain boundary conditions. So, although we can approach any absolutely continuous function via approximating functions from the space of continuously differentiable functions, one can never get close to the infimum of the functional using those approximations. This phenomenon is called Lavrentiev's phenomenon.

More precisely, we record

Property $\Lambda$ (Lavrentiev). With $N \in \{1, 2, \ldots\}$, let $\Omega \subset \mathbb{R}^N$ be an open bounded set; $\Gamma \subset \partial \Omega$ be a boundary subset; and let the functions $L : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $b : \Gamma \to \mathbb{R}$ be given. Set

$$A_1 := \{u \in W^{1,1}(\Omega; \mathbb{R}) \mid u|_{\Gamma} = b\}$$

and

$$A_\infty := \{u \in W^{1,\infty}(\Omega; \mathbb{R}) \mid u|_{\Gamma} = b\}.$$

Define the functional $J : W^{1,1}(\Omega; \mathbb{R}) \to \mathbb{R}$ by

$$J[u] := \int_{\Omega} L(x, u(x), \nabla u(x)) \, dx.$$

We will say that $J$ has property $\Lambda$ if

$$\inf_{u \in A_1} J[u] < \inf_{u \in A_\infty} J[u].$$

Notice that the class of functions $A_\infty$ is dense in the class $A_1$. If, however, the functional $J$ has property $\Lambda$, then the infimum for $J$ over $A_1$ can not be approached using approximations from $A_\infty$. When a functional $J$ has property $\Lambda$, it exhibits the Lavrentiev phenomenon.

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Numerous examples of such functionals have been published since Lavrentiev's first. In 1934, B. Mañá [10] improved Lavrentiev's example; the Lagrangian $L$ was a simple polynomial. In 1984, J. Ball and V. Mizel [2] gave an example with an $L$ that was strictly convex with respect to its third argument. V. Mizel and A. Heinricher [5], in 1988, provided one dimensional constructions with property $\Lambda$; for these examples, they were also able to provide explicit expressions for the pseudo-minimizers and a perturbation method that makes the integrand $L$ strictly convex with respect to its third argument. G. Alberti and P. Majer [1] provided a two dimensional example with an autonomous integrand in 1994. An example without boundary conditions was found by K. Dani, W. Hrusa and V. Mizel [4] in 1996. W. Hrusa [7] also created an example in one dimension where the functional depends only on $u$, $u'$ and $u''$; that is $J$ has the form $\int_a^b L(u(x), u'(x), u''(x)) \, dx$, where $u : (a, b) \to \mathbb{R}$. The diversity of examples in the literature demonstrate that the Lavrentiev phenomenon can appear in a wide variety of variational problems; in fact, recent work by W. Hrusa and V. Mizel strongly suggests, at least for one dimensional problems, that the phenomenon is generic.

One question that has arisen concerns the possible sensitivity of the infimum for $J$ to the regularity of the class of admissible functions. More precisely: given a functional $J$ with property $\Lambda$, if for every $p \in [1, \infty]$, we define the classes

$$\mathcal{A}_p := \{ u \in W^{1,p}(\Omega; \mathbb{R}) \mid u|_{\Gamma} = b \},$$

and define the function $I : [1, \infty] \to \mathbb{R}$ by

$$I : p \mapsto \inf_{u \in \mathcal{A}_p} J[u],$$

how might $I$ depend on the Sobolev exponent $p$? This is the question we consider in this report.

Until recently, for all examples of the Lavrentiev phenomenon, there existed some $p^* \in (1, \infty)$ and constants $c_0, c_1 \in \mathbb{R}$ such that $\forall p_0 \in [1, p^*)$ and $\forall p_1 \in [p^*, \infty]$

$$c_0 = I(p_0) < I(p_1) = c_1.$$

Of course, it is a simple matter of addition to construct examples with any countable number of $p_i$'s and constants $c_i$ so that $\forall p_0 \in [1, p_0^*)$ and $\forall p_i \in [p_i^*, p_{i+1}^*)$

$$c_0 = I(p_0) < c_1 = I(p_1) < \cdots < c_i = I(p_i) < \cdots.$$

In other words, all the examples had an increasing function $I$ that was right continuous and piecewise constant, with at most a countable number of steps. In 1999, however W. Hrusa and A. Segal [12], constructed one dimensional examples of $J$, with smooth integrands, such that

$$I(p) = \begin{cases} c_1, & 1 \leq p < p^*; \\ c_2, & p = p^*; \\ c_3, & p^* < p \leq \infty; \end{cases}$$

where $p^* \in [1, \infty]$ and $c_1, c_2, c_3 \in \mathbb{R}$ such that $c_1 < c_2 < c_3$. These examples show that $I$ can have a jump discontinuity at any $p^* \in [1, \infty]$ and that $I$ may be right continuous, left continuous or neither.

In this report, we will provide examples where $I$ is an absolutely continuous function. Moreover, we will show that, given $p_0, p_1 \in (1, \infty)$ and $c \in [0, \infty)$, if $\alpha$ is a monotone absolutely continuous function such that

$$\alpha(p) = 0 \quad \forall p \in [1, p_0),$$

and
and

\[ \alpha(p) = c \quad \forall p \in [p_1, \infty), \]

then there is a functional \( J \) such that for each \( p \in [1, \infty] \) \( I(p) = \alpha(p) \). So, the infima for our examples will have a prescribed continuous dependence upon the exponent of the Sobolev space from which the admissible functions are taken. As the exponent decreases and the class of admissible functions expands, we are able to more closely approach the absolute infimum of the functional. This absolute minimum, however, can never be attained until the exponent of the Sobolev space decreases to at least \( p_1 \).

2. The Basic Idea

The underlying idea for all the examples presented in this report is very simple. Unfortunately, the proofs we have for these examples rely on a lot of computation, which seems to make everything that is being done appear more complicated than it really is. It is hoped that after reading this section, one will have the basic idea of what we will prove and a confidence in the correctness of the results.

First, let us extract a result from [5] upon which the idea is based. For each \( q \in (1, \infty) \), define \( L(\cdot, \cdot; q) : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by

\[ L(\cdot, \cdot; q)(x, u, z) \mapsto (|u|^q - x)^{\frac{1}{q-1}}, \]

and let the functional \( J[\cdot; q] : W^{1,1}((0,1); \mathbb{R}) \to \mathbb{R} \) be given by

\[ J[\cdot; q] : [u] \mapsto \int_{(0,1)} L(x, u(x), u_x(x); q) \, dx. \]

Also, for each \( p \in [1, \infty] \), let us set

\( A_p := \{ u \in W^{1,p}((0,1); \mathbb{R}) \mid u(0) = 0 \text{ and } u(1) = 1 \}. \)

In [5], it was shown that, for each \( q \in (1, \infty) \), the functional \( J[\cdot; q] \) exhibits the Lavrentiev phenomenon. By using field theory and the invariance of \( L \) under a certain rescaling, it was proven that

\[ 1 \leq p < \frac{1}{q-1} \implies \inf_{u \in A_p} J[u; q] = J[x^{\frac{1}{q}}; q] = 0, \]

while

\[ \frac{1}{q-1} \leq p \implies \inf_{u \in A_p} J[q][u] = J[x^{\frac{3}{q+1}}; q] = \frac{2}{3} \left( \frac{q-1}{q+1} \right)^3 > 0. \]

We see that the absolute minimum is attained in Sobolev spaces with exponent less than \( \frac{1}{q-1} \), but this minimum is unattainable when the space of admissible functions is taken from the Sobolev spaces with exponent greater than \( \frac{1}{q-1} \). So, there is a jump in the infimum for \( J \) as the Sobolev exponent increases past \( \frac{1}{q-1} \).

For us, the particularly important part of this result is that it gives us the exact value for the infimum of the functional in the different Sobolev spaces. Moreover, the result provides us with the functions that give us these different infima. Let us define \( I(\cdot; q) : [1, \infty) \to \mathbb{R} \) as

\[ I(\cdot; q) : p \mapsto \inf_{u \in A_p} J[u; q]. \]
From [5], we have an explicit formula for $I$: for each $q \in (1, \infty)$

$$I(p; q) = \begin{cases} 
0, & 1 \leq p < \frac{1}{q-1}; \\
\frac{2}{3} \left( \frac{q-1}{q+2} \right)^{\frac{3q}{q+1}} \frac{1}{q-1} \leq p. 
\end{cases}$$

Let us also parametrize the functions that give us each of the above infima. For any $q \in (1, \infty)$, we define $u_{am}(\cdot; q) : (0, 1) \to \mathbb{R}$ and $u_{pm}(\cdot; q) : (0, 1) \to \mathbb{R}$ as

$$u_{am}(\cdot; q) : x \mapsto x^{\frac{1}{q}}$$

and

$$u_{pm}(\cdot; q) : x \mapsto x^{\frac{3}{q+1}}.$$ 

So, we can now write

$$I(p; q) = \inf_{u \in A_p} J[u; q] = \begin{cases} 
J[u_{am}(\cdot; q); q], & 1 \leq p < \frac{1}{q-1}; \\
J[u_{pm}(\cdot; q); q], & \frac{1}{q-1} \leq p. 
\end{cases}$$

Now, the essential idea for the examples in this report is to add a dimension to the domain of the admissible functions and use this extra dimension to parametrize $q$. For the moment, we will set the domain of the admissible functions to be the square

$$\Omega := (0, 1) \times (0, 1).$$

Also, set

$$A_p := \left\{ W^{1,p}(\Omega; \mathbb{R}) : u(0, \cdot) = 0 \text{ and } u(1, \cdot) = 1 \right\},$$

and let $q : (0, 1) \to (1, \infty)$ be some given continuous function. We define a new functional $J : W^{1,1}(\Omega, \mathbb{R}) \to \mathbb{R}$ by

$$J : [u] \mapsto \int_{(0, 1)} J[u(\cdot, y); q(y)] dy = \int_{\Omega} L(x, u(x, y), u_y(x, y); q(y)) dxdy.$$ 

Defining $I : [1, \infty) \to \mathbb{R}$ as

$$I : p \mapsto \inf_{u \in A_p} J[u],$$

one might suspect that

$$I(p) = \int_{(0, 1)} I(p; q(y)) dy$$

and that the minimum is attained by the mapping $u^* : \Omega \to \mathbb{R}$ given by

$$u^* : (x, y) \mapsto \begin{cases} 
u_{am}(x; q(y)), & 1 \leq p < \frac{1}{q(y)-1}; \\
u_{pm}(x; q(y)), & \frac{1}{q(y)-1} \leq p. 
\end{cases}$$
If (3) were in \( A_p \) and \( q(\cdot) \) were non-constant, then (2) would imply that we have an example of the Lavrentiev phenomenon where the infimum increases continuously with the exponent of the Sobolev space.

The mapping given in (3), however, suffers a jump discontinuity whenever \( \frac{1}{q(y)} \) increases past \( p \). Therefore (3) is not a member of the admissible space \( A_p \) in general. What we can do though is construct a sequence, using members from \( A_p \), which approaches a.e. to the mapping given in (3). This sequence allows us to conclude

\[
I(p) \leq \int_{(0,1)} I(p; q(y)) \, dy.
\]

Meanwhile the results in [5] show that

\[
I(p) \geq \int_{(0,1)} I(p; q(y)) \, dy.
\]

So, it follows that the guess in (2) is correct.

This is the main idea behind the results and proofs that will be presented in this report. We will not use the functional \( J \) given in (1) though. Instead, we will use a modified version of this functional. In addition to having properties similar to those of (1), this modified functional will provide us with greater control on the regularity of its Lagrangian and how its infimum increases with the Sobolev exponent.

3. The Functional

In this section we construct the functionals we wish to examine. A simple result regarding the regularity of our Lagrangians is also discussed.

Whenever it is convenient, we identify an equivalence class with any one of its representatives.

First, we define more precisely the subclass of the monotone absolutely continuous functions that was mentioned in the introduction. These functions will be used to prescribe how the infimum of our functionals depend upon the Sobolev exponent. For each \( p_0, p_1 \in (1, \infty) \) such that \( p_0 < p_1 \), set

\[
\mathcal{M}_{p_0, p_1} := \left\{ \alpha \in W^{1,1}_{\text{loc}}([1, \infty]; [0, \infty)) \mid \text{conditions (i)-(iii) below are satisfied} \right\}.
\]

Conditions (i)-(iii) are the following:

(i) \( \alpha_{|[1, p_0]} = 0 \);
(ii) \( \alpha_{p_{p_0, p_1}} \geq 0 \);
(iii) \( \alpha_{p_{p_1, \infty}} = 0 \).

We also put

\[
\mathcal{M} := \bigcup_{\substack{p_0 > 0, \\ p_1 > p_0}} \mathcal{M}_{p_0, p_1}.
\]

If \( \alpha \in \mathcal{M}_{p_0, p_1} \), then it must be an absolutely continuous nondecreasing function that is 0 on \([1, p_0]\) and constant on \([p_1, \infty]\). Given an \( \alpha \in \mathcal{M} \), we will construct a functional whose infimum duplicates \( \alpha \) as the Sobolev exponent \( p \) increases from 1 to \( \infty \).

In what follows, we assume \( p_0, p_1 \in (1, \infty) \) have been selected such that \( p_0 < p_1 \) and \( \alpha \in \mathcal{M}_{p_0, p_1} \) has been given. For every \( q \in (1, \infty) \) and \( m \in [3q, \infty) \), we let
\[ K(\cdot, \cdot; q, m) : (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \] and \[ L(\cdot, \cdot; q, m) : (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \] be defined by

\[ K(\cdot, \cdot; q, m) : (x, u, z; q, m) \mapsto |u|^{\frac{m-1}{m-2}} |u^{\frac{q}{m-2}} - x|^m \]

and

\[ L(\cdot, \cdot; q, m) : (x, u, z; q, m) \mapsto [\alpha_p(q)][s(q, m)][K(x, u, z; q, m)], \]

where we have put

\[ s(q, m) := \left( \frac{m-3}{2} \right) \left[ \frac{m-1}{m} q \right]^m. \]

Later we will see that if we were to use the mapping \( K \) as a Lagrangian, then under certain boundary conditions we would have a one-dimensional variational problem that exhibits the Lavrentiev phenomenon: the infimum for the problem is zero if we take the competing functions from \( W^{1,1} \), but it jumps to some positive value if we require the admissible functions to be in \( W^{1,q} \). The factor \( s \) just normalizes the magnitude of this jump to 1. So, if we use \( L \) as a Lagrangian, then the infimum will be either 0 or \( \alpha_p(q) \) depending on whether or not the exponent of the Sobolev space is less than \( q \).

As was described in the previous section, the domain for the competing functions is two-dimensional, and one of these dimensions will be used to parametrize the \( q \) and \( m \) in (5). For each \( y, \bar{y} \in \mathbb{R} \) such that \( y < \bar{y} \), set

\[ \Omega_{y, \bar{y}} := (0, 1) \times (y, \bar{y}). \]

Now we parametrize \( q \) and \( m \). Let \( q, m : (y, \bar{y}) \to (1, \infty) \) be given continuous functions such that \( \forall y \in (y, \bar{y}) m(y) \geq 3q(y) \). The functional \( J : W^{1,1}(\Omega_{y, \bar{y}}; \mathbb{R}) \to \mathbb{R} \) is then given by

\[ J : [u] \mapsto \int_{\Omega_{y, \bar{y}}} L(x, u, u_x; q(y), m(y)) \, dx dy. \]

If we were to require the admissible functions to belong to \( W^{1,p} \), as well as satisfy certain boundary conditions, then we would be able to show that the infimum for \( J \) is

\[ \int_{\{y \mid q(y) \leq p\}} \alpha_p(q(y)) \, dy. \]

However, we only wish to construct some functional with an infimum that matches \( \alpha \).

So, rather than proving the above statement for general parametrizations of \( q \) and \( m \) we will make a particular choice. We put \( y = p_0 \), \( \bar{y} = p_1 \) and \( q(y) = y \), and we only restrict the parameter \( m \) to be some constant so that \( m \geq 3p_1 \). So for \( m \in [3p_1, \infty) \), the functional \( J[\cdot; m] : W^{1,1}(\Omega_{p_0, p_1}; \mathbb{R}) \to \mathbb{R} \) that we wish to examine is defined as

\[ J[\cdot; m] : [u] \mapsto \int_{\Omega_{p_0, p_1}} L(x, u, u_x; y, m) \, dx dy, \]

where \( L \) was given in (5). As spaces of admissible functions, we set

\[ \mathcal{A}_p := \{ u \in W^{1,p}(\Omega_{p_0, p_1}; \mathbb{R}) \mid u(0, \cdot) = 0 \text{ and } u(1, \cdot) = 1 \}. \]
For convenience, we also define $I(\cdot ; m) : [1, \infty) \to \mathbb{R}$ by

$$I(\cdot ; m) : p \mapsto \inf_{u \in A_p} J[u; m].$$

Our objective then is to show that

$$\forall m \in [3p_1, \infty), \forall p \in [1, \infty] \quad I(p; m) = \int_{\{q \leq p\}} \alpha_p(y) \, dy = \alpha(p).$$

Other than requiring it to be a constant, we have not fixed the parameter $m$; via this parameter, we will have some control on the regularity of the Lagrangian.

More precisely we have the following:

**Remark 1.** Choose an even integer $m$ such that $m \in [3p_1, \infty)$. Put $k = [\frac{m - 3p_1}{p_1 - 1}]$, the smallest integer at least as large as $\frac{m - 3p_1}{p_1 - 1} - 1$. The following is true at each $(x, y, u, z) \in \Omega_{p_0, p_1} \times \mathbb{R} \times \mathbb{R}$: for any non-negative integers $i_x, i_y, i_u$ and $i_z$, with $0 \leq i_u \leq k$

$$\frac{\partial^{i_x} \partial^{i_y} \partial^{i_u} \partial^{i_z}}{\partial^{i_x} u \partial^{i_y} y \partial^{i_u} u \partial^{i_z} z} [s(y, m)K(x, u, z; y, m)]$$

is continuous with respect to $x, y, u$ and $z$.

The truth of this remark can be easily seen by writing

$$s(y, m)K(x, u, z; y, m) = \frac{(m-3)(m-2)}{2} \left( \frac{m-1}{m} \right)^m \left[ \frac{y}{y-1} \right]^m \left[ |u|^{\frac{m-2}{y-1}} + 2|x|^{\frac{m-3}{y-1}} + x^2 |u|^{\frac{m-2}{y-1}} \right] z^m$$

and observing that each of the quantities $\frac{m-2}{y-1}, \frac{m-3}{y-1}$ and $\frac{m-2}{y-1}$ is larger than $k$ and that $\frac{1}{y-1}$ is $C^\infty$ on $\Omega_{p_0, p_1}$. So, the mapping $s(y, m)K(x, u, z; y, m)$ is a $C^\infty$ function with respect to $x, y$ and $p$ and a $C^k$ function with respect to $u$. The regularity with respect to $u$ may be improved by taking $m$ to be a larger even integer. Based upon this remark and the definition of $L$ (5), we see that the smoothness of $L(x, u, z; y, m)$ is essentially determined by the smoothness of $\alpha_p(y)$ over the interval $(p_0, p_1)$.

For an example of one of the functionals we have constructed above, let us put $p_0 = \frac{5}{2}$ and $p_1 = \frac{5}{2}$. We take $\alpha \in M_{p_0, p_1}$ to be

$$\alpha : p \mapsto \begin{cases} 0, & 0 \leq p < \frac{3}{2}; \\ \frac{3}{2} \leq p < \frac{5}{2}; \\ 1, & \frac{5}{2} \leq p. \end{cases}$$

So $\alpha_p = 1$ over the interval $(\frac{3}{2}, \frac{5}{2})$. By setting $m = 14$, our Lagrangian becomes

$$L(x, u, z; y, 14) = 66 \left( \frac{3}{14} \right)^{14} \left( \frac{y}{y-1} \right)^{14} |u|^{\frac{14+2k}{y-1}} (|u|^{y-1} - x)^2 z^{14}.$$
4. $I(p; m) \geq \alpha(p)$

The goal for this section is to prove

**Lemma 1.** Let $m \in [3p_1, \infty)$. For each $p \in [1, \infty]$ we have $I(p; m) \geq \alpha(p)$.

**Proof.** The proof is split into three parts. The first case is for $1 \leq p \leq p_0$. In the second case, we prove the lemma for $p_0 < p < p_1$. The last case covers $p_1 \leq p$.

**Case 1.** $1 \leq p \leq p_0$:

If $p \in [1, p_0]$, then $\alpha(p) = 0$ by the definition of $\mathcal{M}_{p_0,p_1}$. Since $L \geq 0$, we have

$$I(p; m) = \inf_{u \in \mathcal{A}_p} \int_{\Omega_{p_0,p_1}} L(x, u, u; y, m) \, dx \, dy \geq 0.$$ 

Therefore $I(p; m) \geq \alpha(p)$.

For the remaining two cases, we will reduce our problem to a collection of one-dimensional problems. To this end, it will be useful to define one-dimensional analogues of the classes $\mathcal{A}_p$. So for every $p \in [1, \infty]$, we set

(12) $B_p := \{ v \in W^{1,p}((0,1); \mathbb{R}) \mid v(0) = 0 \text{ and } v(1) = 1 \}$.

We now proceed to Case 2.

**Case 2.** $p_0 < p < p_1$:

Of the three cases, this is the most complicated. The idea is to treat the two-dimensional problem as a one-dimensional variational problem at each $y \in (p_0, p]$. A lower bound is then established for each of these one-dimensional problems, and this will lead to the desired lower bound for the two-dimensional problem.

Let $u \in \mathcal{A}_p$ be given. Recall

$$J[u; m] = \int_{\Omega_{p_0,p_1}} L(x, u, u; y, m) \, dx \, dy$$

$$= \int_{\Omega_{p_0,p_1}} \alpha_p(y) s(y, m) K(x, u, u; y, m) \, dx \, dy$$

and $sK$ and $\alpha_p$ are nonnegative by (6) and (4). Using Fubini's theorem, the definition of $\Omega_{p_0,p_1}$ (7) and the fact that for each $y \in (p_0, p] \mathcal{A}_p \subseteq \mathcal{A}_y$, we may write

$$J[u; m] = \int_{\{ p_0, p_1 \}} \alpha_p(y) s(y, m) \int_{(0,1)} K(x, u, u; y, m) \, dx \, dy$$

$$+ \int_{\{ y | p_0 < y \leq p \}} \alpha_p(y) s(y, m) \int_{(0,1)} K(x, u, u; y, m) \, dx \, dy$$

$$\geq \int_{\{ y | p_0 < y \leq p \}} \alpha_p(y) s(y, m) \int_{(0,1)} K(x, u, u; y, m) \, dx \, dy.$$
Since \( u \in W^{1,p}(\Omega_{p_0,p}; \mathbb{R}) \), for almost every \( y \in (p_0, p] \) we have \( u(\cdot, y) \in B_p \subset B_y \). It follows that

\[
J[u; m] \geq \int_{\{y \mid p_0 < y \leq p\}} \alpha_p(y) s(y, m) \left[ \inf_{v \in B_p} \int_{(0,1)} K(x, \tilde{v}, \tilde{v}_z; y, m) \, dx \right] \, dy
\]

(13)

\[
\geq \int_{\{y \mid p_0 < y \leq p\}} \alpha_p(y) s(y, m) \left[ \inf_{v \in B_p} \int_{(0,1)} K(x, \tilde{v}, \tilde{v}_z; y, m) \, dx \right] \, dy.
\]

Let us define \( G : W^{1,1}((0,1); \mathbb{R}) \rightarrow L^1((p_0, p); \mathbb{R}) \) by

(14)

\[
G : [\tilde{v}] \mapsto \int_{(0,1)} K(x, \tilde{v}, \tilde{v}_z; y, m) \, dx.
\]

Now (13) becomes

(15)

\[
J[u; m] \geq \int_{\{y \mid p_0 < y \leq p\}} \alpha_p(y) s(y, m) \left[ \inf_{v \in B_p} G[\tilde{v}](y) \right] \, dy.
\]

It is clear that for the proof of this case, it suffices to show for almost every \( y \in (p_0, p] \) that \( \inf_{v \in B_p} G[\tilde{v}](y) \) is bounded below by \( \frac{1}{s(y, m)} \). We will treat this as a one-dimensional variational problem at each \( y \in (p_0, p] \) and use a field theory argument analogous to the one used in [5]. The key ingredients for this argument are the solutions to the Euler-Lagrange equations for the functional \( G[\cdot](y) \) and the convexity of \( K \) with respect to \( z \).

We fix \( y \in (p_0, p] \) for now. To find the solutions of these Euler-Lagrange equations, we use the following homogeneity property of \( K \): for any \( \varepsilon \in \mathbb{R} \)

(16)

\[
K(e^\varepsilon x, e^{\frac{\varepsilon}{\gamma}} u, e^{\frac{-\varepsilon}{\gamma}} z; y, m)e^\varepsilon = K(x, u, z; y, m).
\]

This property gives us a variational symmetry for \( G[\cdot](y) \). A theorem of E. Noether's tells us this symmetry has a corresponding conservation law [11]. The conservation law provides a means of reducing the order of the Euler-Lagrange equations by one. One way of finding the conservation law (check for [9]) is to observe that for every \( \varepsilon \in \mathbb{R} \) (at \( \varepsilon = 0 \) in particular) we have

\[
\frac{\partial}{\partial \varepsilon} \left[ K(e^\varepsilon x, e^{\frac{\varepsilon}{\gamma}} u, e^{\frac{-\varepsilon}{\gamma}} v; y, m)e^\varepsilon \right] = 0.
\]

It follows that for every \( v \in B_y \)

(17)

\[
xK_{,x} + \frac{(y-1)v}{y} K_{,u} - \frac{v}{y} K_{,z} + K = 0.
\]

Suppose \( v \in C^2((0,1); \mathbb{R}) \). Then we can use (17) to write

(18)

\[
\frac{d}{dx} \left[ xK + \frac{(y-1)v}{y} K_{,z} - xv_{,x} K_{,z} \right] = \left( K_{,u} - \frac{d}{dx} K_{,z} \right) \left( xv_{,x} - \frac{(y-1)v}{y} \right).
\]

Now, let us further suppose that \( v \) satisfies the Euler-Lagrange equations

(E-L)

\[
K_{,u} - \frac{d}{dx} K_{,z} = 0.
\]
Then (18) reduces to
\[
(19) \quad xK + \frac{(y - 1)v}{y} K_{z} - xv_xK_z = C,
\]
where \(C\) is a constant. This is the conservation law corresponding to the variational symmetry (16). It is clear that if (19) holds, then (18) implies that either the Euler-Lagrange equations (E-L) are satisfied or \(v(x) = \lambda x \frac{y-1}{y}\), for some \(\lambda \in \mathbb{R}\). We will find, though, that if \(v\) satisfies (19) with \(C = 0\), then \(v(x) \neq \lambda x \frac{y-1}{y}\) for any \(\lambda \in \mathbb{R}\backslash\{0\}\). It follows that \(v\) satisfying (19) with \(C = 0\) must also satisfy (E-L).

Let us consider (19) with \(C = 0\). We will look for \(v\) such that
\[
(20) \quad xK + \frac{(y - 1)v}{y} K_z - xu_xK_z = 0.
\]
From definition (4) for \(K\), we find
\[
K_z = \frac{m}{z} K_{z}.
\]
So the conservation law (20) can be reformulated as
\[
(21) \quad K \left[ \frac{(y - 1)m}{y} \frac{v}{v_x} - (m - 1)x \right] = 0
\]
We now assume that \(v_x \neq 0\) on \((0, 1)\). Then, since \(v(x) \neq \lambda x \frac{y-1}{y}\), we must have \(K \geq 0\). So (21) simplifies to
\[
\frac{v}{v_x} = \frac{(y - 1)m y}{(m - 1)y x}.
\]
Thus for any \(\lambda \in \mathbb{R}\)
\[
(22) \quad v(x; \lambda) := \lambda x \frac{(x-1)m}{(m-1)y}
\]
is a solution to (19). As argued above, the mapping \(v(\cdot; \lambda)\) must also satisfy the Euler-Lagrange equations (E-L) for each \(\lambda \in \mathbb{R}\).

Of course, only \(v(\cdot; 1)\) is actually a member of \(B_y\), but we are going to use all the solutions we have found to construct a point-slope field. Observe that the family of curves \(v(\cdot; \lambda)\) covers the slab \((0, 1) \times \mathbb{R}\) simply. So for each point in \((0, 1) \times \mathbb{R}\) there is exactly one \(\lambda \in \mathbb{R}\) such that \(v(\cdot; \lambda)\) passes through this point. In fact, by defining \(\lambda : (0, 1) \times \mathbb{R} \to \mathbb{R}\) as
\[
(23) \quad \lambda : (x_0, w_0) \mapsto \frac{(1-x_0)w_0}{m(w_0 - 1)\frac{y}{x}}
\]
we have
\[
v(x_0; \lambda(x_0, w_0)) = w_0.
\]
Now, we may assign to each point of this slab the slope, at the point, of the unique curve \(v(\cdot; \lambda)\) passing through this point. We define \(\pi : (0, 1) \times \mathbb{R} \to \mathbb{R}\) as
\[
\pi : (x_0, w_0) \mapsto \left. \frac{d}{dx} v(x; \lambda(x_0, w_0)) \right|_{x=x_0} = \frac{(y - 1)m w_0}{(m - 1)y x_0}.
\]
The mapping \(\pi\) constitutes the point-slope field. Notice that \(\pi\) is \(C^\infty_{\text{loc}}\) on \((0, 1) \times \mathbb{R}\).
Since we constructed the mapping $\pi$ using the solutions of the Euler-Lagrange equations and $\pi$ is differentiable away from the line $\{0\} \times \mathbb{R}$, it follows that the mapping $K^* : (0, 1) \times W^{1, y}((0, 1); \mathbb{R}) \to \mathbb{R}$ given by

$$K^* : (x, w) \mapsto K(x, w(x), \pi(x, w(x)); y, m) + [w(x) - \pi(x, w(x))]K_x(x, w(x), \pi(x, w(x)); y, m)$$

is an integrand for a path independent Hilbert integral \cite{3, 6}, so long as $x$ remains away from zero. In other words, for each $x_0 \in (0, 1)$ the integral

$$\int_{[x_0, 1]} K^*(x, V) \, dx$$

can be extended to all parametric curves $V$ in $(x_0, 1) \times \mathbb{R}$ that have piecewise continuous derivatives, and the value of the integral will depend only upon the endpoints of the curve. This property is vital to our proof.

Another property that we will need is the convexity of $K$ with respect to the $z$ argument. This convexity implies that the Weirstrass excess function is nonnegative. So, we have

$$E(x, u, z, z') := K(x, u, z; y, m) - K(x, u, z'; y, m) - (z - z')K_z(x, u, z'; y, m) \geq 0.$$  

We are now prepared to compare the values of $G[\nu(\cdot; 1)](y)$ with the value of $G[\cdot](y)$ for other competing functions. Let $\tilde{v} \in \mathcal{B}_y$ and $\sigma \in (0, 1)$ be given. As in \cite{5}, we use the independence of the Hilbert integral and the nonnegativity of the Weirstrass excess function to write

$$G[\tilde{v}] (y) = \int_{(\sigma, 1)} [K(x, \tilde{v}, \tilde{v}, y, m) - K^*(x, \tilde{v}) + K^*(x, \tilde{v})] \, dx + \int_{(0, \sigma)} K(x, \tilde{v}, \tilde{v}, y, m) \, dx$$

$$\geq \int_{(\sigma, 1)} E(x, \tilde{v}, \tilde{v}, x, \pi(x, \tilde{v}); y, m) \, dx + \int_{(\sigma, 1)} K^* (x, \tilde{v}) \, dx$$

$$\geq \int_{(\sigma, 1)} K^* (x, \tilde{v}) \, dx$$

$$= \int_{(\sigma, 1)} K^* (x, v(x; 1)) \, dx - \int_{(0, \sigma)} K^* (x, v(x; \lambda(\sigma, \tilde{v}(\sigma)))) \, dx.$$  

Since $\tilde{v} \in W^{1, y}((0, 1); \mathbb{R})$, Hölder’s inequality implies

$$\lim_{x \to 0} \frac{\tilde{v}(x)}{x^{\frac{1}{y' - p}}} = 0.$$  

It follows from this that the second integral in (24) approaches zero as $\sigma \to 0$. So

$$\inf_{\tilde{v} \in \mathcal{B}_y} G[\tilde{v}] (y) \geq G[\nu_1] (y) = s(y, m).$$

Recalling (15)

$$J [u, m] \geq \int_{\{y \mid p < y \leq p\}} \alpha_p (y) s(y, m) \left[ \inf_{\tilde{v} \in \mathcal{B}_y} G[\tilde{v}] (y) \right] \, dy,$$
we may now write

$$J[u; m] \geq \int_{\{y \mid p_0 < y \leq p\}} \alpha_p(y) \, dy.$$  

Therefore $I(p; m) \geq \alpha(p)$ and the lemma has been proven for this case. There remains only one case to prove.

**Case 3.** $p_1 \leq p$:

Since $p \geq p_1$ and $\alpha \in M_{p_0, p_1}$, we have $\alpha(p) = \alpha(p_1)$. Using what was done in case 2 above, we find

$$J[u; m] \geq \int_{\{y \mid p_0 < y \leq p_1\}} \alpha_p(y) s(y, m) \left[ \inf_{\vec{v} \in B_k} \int_{(0, 1)} K(x, \vec{v}, \vec{v}_x; y, m) \, dx \right] \, dy$$

$$\geq \int_{\{y \mid p_0 < y \leq p_1\}} \alpha_p(y) \, dy$$

$$= \alpha(p_1).$$

Again, we have $I(p; m) \geq \alpha(p)$ and the lemma is proven. \(\square\)

In this section, we have shown $I(p; m) \geq \alpha(p)$. In the next section we will prove the opposite inequality.

5. $I(p; m) \leq \alpha(p)$

We wish to prove

**Lemma 2.** Let $m \in [3p_1, \infty)$. For each $p \in [1, \infty]$ we have $I(p; m) \leq \alpha(p)$

**Proof.** As in Lemma (1), the proof is split into three cases. The first of which is $1 \leq p \leq p_0$; the second $p_0 < p < p_1$; and the third $p_1 \leq p$.

**Case 1.** $1 \leq p \leq p_0$:

For this case, since $p \leq p_0$ we may simply choose $u \in A_p$ given by

$$u(x, y) := x^{\frac{p-1}{q}}.$$

Clearly $J[u; m] = 0 = \alpha(p)$ since $\alpha \in M_{p_0, p_1}$. Therefore $I(p; m) \leq \alpha(p)$.

To prove the remaining cases, for each $p \in (p_0, \infty]$ we will construct a sequence \(\{u^{(n)}\}_{n=1}^\infty \subset A_p\) such that \(\lim_{n \to \infty} J[u^{(n)}] = \alpha(p)\). Again, it will follow that $I(p; m) \leq \alpha(p)$.

**Case 2.** $p_0 < p < p_1$:

Put $N = \left[ \frac{1}{p - 1} \right]$. For each $n \in \{N, N + 1, \ldots\}$ we will divide the domain $\Omega_{p_0, p_1}$ into 5 subdomains and construct a sequence of mappings \(\{u^{(n)}\}_{n=N}^\infty \subset A_p\) over $\Omega_{p_0, p_1}$ using these subdomains. The sequence we construct will have the properties we need to prove the lemma for this case.
For each \( n \in \{N, N + 1, \cdots \} \), set

\[
\Omega_1^{(n)} := \{(x,y) \in \mathbb{R}^2 \mid p + \frac{1}{n} < y < p_1 \text{ and } 0 < x < 1 \};
\]

\[
\Omega_2^{(n)} := \{(x,y) \in \mathbb{R}^2 \mid p \leq y \leq p + \frac{1}{n} \text{ and } 0 < x \leq p - y + \frac{1}{n} \};
\]

\[
\Omega_3^{(n)} := \{(x,y) \in \mathbb{R}^2 \mid p \leq y \leq p + \frac{1}{n} \text{ and } p - y + \frac{1}{n} < x < 1 \};
\]

\[
\Omega_4^{(n)} := \{(x,y) \in \mathbb{R}^2 \mid p_0 < y < p \text{ and } 0 < x \leq \frac{1}{n} \};
\]

and

\[
\Omega_5^{(n)} := \{(x,y) \in \mathbb{R}^2 \mid p_0 < y < p \text{ and } \frac{1}{n} < x < 1 \};
\]

also put

\[
r^{(n)}(y) := \left(\frac{p-1}{m-1}p + n \frac{p+\frac{1}{n} - 1}{p+\frac{1}{n}} - \frac{(p-1)n}{(m-1)p} \right) (y - p).
\]

For each \( n \in \{N, N + 1, \cdots \} \), the union \( \bigcup_{1 \leq i \leq 5} \Omega_i^{(n)} = \Omega_{p_0,p_1} \), and the function \( r^{(n)} \) is a linear mapping such that \( r^{(n)}(p) = \left(\frac{p-1}{m-1}p \right) \) and \( r^{(n)}(p + \frac{1}{n}) = \left(\frac{p+\frac{1}{n}-1}{p+\frac{1}{n}} \right) \).

Now, we define the sequence \( \{u^{(n)}\}_{n=N}^{\infty} \subset \mathcal{A}_p \) by

\[
u^{(n)}(x,y) := \begin{cases}
    x^\frac{p-1}{p}, & (x,y) \in \Omega_1^{(n)}; \\
    (p - y + \frac{1}{n}) v^{(n)}(y)-1 x, & (x,y) \in \Omega_2^{(n)}; \\
    x^{r^{(n)}(y)}, & (x,y) \in \Omega_3^{(n)}; \\
    \left(\frac{p-1}{m-1}p \right) x, & (x,y) \in \Omega_4^{(n)}; \\
    x^{\frac{(p-1)n}{(m-1)p}}, & (x,y) \in \Omega_5^{(n)}.
\end{cases}
\]

It is straightforward to verify that \( \{u^{(n)}\}_{n=N}^{\infty} \) is in fact a subset of \( \mathcal{A}_p \) at each point in the set \( \bigcup_{1 \leq i \leq 4} \Omega_i^{(n)} \); the exponent on \( x \) is always greater than \( \frac{p-1}{p} \).

Now, for each element of this sequence, we find

\[
J[u^{(n)};m] = \int_{\Omega_{p_0,p_1}} L(x,u^{(n)};u^{(n)}_{x};y,m) \, dx \, dy
\]

\[
= \int_{\Omega_{p_0,p_1}} L(x,u^{(n)};u^{(n)}_{x};y,m) \, dx \, dy
\]

\[
= \int_{\Omega_2^{(n)}} L(x,u^{(n)};u^{(n)}_{x};y,m) \, dx \, dy + \int_{\Omega_3^{(n)}} L(x,u^{(n)};u^{(n)}_{x};y,m) \, dx \, dy
\]

\[
+ \int_{\Omega_4^{(n)}} L(x,u^{(n)};u^{(n)}_{x};y,m) \, dx \, dy + \int_{\Omega_5^{(n)}} L(x,u^{(n)};u^{(n)}_{x};y,m) \, dx \, dy
\]

It is another straightforward computation to show that the first three integrals in (25) go to zero as \( n \to \infty \); one can use, for instance, the following: since \( x \in (0,1) \)

\[
u \leq x^\frac{p-1}{p} \implies R(x,u,z,y,m) \leq 2x^\frac{m}{p},
\]

and for each \( y \in [p, p + \frac{1}{n}] \)

\[
\frac{p-1}{p} \leq r^{(n)}(y) \leq \left(\frac{p-1}{m-1}p \right).
\]
For the last integral in (25), we have
\[ \int_{\Omega_0^n} \int L(x, u^{(n)}_x, u^{(n)}_y, y, m) \, dx \, dy = \int_{\{y | p_0 < y < p\}} \int K(x, u^{(n)}_x, u^{(n)}_y; y, m) \, dx \, dy. \]
At each \( y \in (p_0, p) \) we see that \( u^{(n)}_x = v_1 \), where \( v_1 \) was the solution to the Euler-Lagrange equations that we found earlier (22). So as \( n \to \infty \), the inner integral in (26) approaches \( \frac{1}{f(y, m)} \) at each \( y \in (p_0, p_1) \). Thus
\[ \lim_{n \to \infty} J[u^{(n)}] = \int_{\{y | p_0 < y < p\}} \alpha_p(y) \, dy = \alpha(p). \]
This implies \( \inf_{u \in \mathcal{A}_p} J[u; m] \leq \alpha(p) \). Therefore \( I(p; m) \leq \alpha(p) \) for this case.

The final case for this lemma is similar but simpler than the previous case.

**Case 3**. \( p_1 \leq y : \)

For this case, we will only divide the domain into two subdomains. For every \( n \in \{1, 2, \ldots \} \), set
\[ \Omega_1^{(n)} := \{(x, y) \in \mathbb{R}^2 \mid p_0 < y < p_1 \text{ and } 0 < x < \frac{1}{n}\} \]
and
\[ \Omega_2^{(n)} := \{(x, y) \in \mathbb{R}^2 \mid p_0 < y < p_1 \text{ and } \frac{1}{n} < x < 1\}. \]
So \( \Omega_1^{(n)} \cup \Omega_2^{(n)} = \Omega_{p_0, p_1} \) for each \( n \in \{1, 2, \ldots \} \).

We let the sequence \( \{u^{(n)}\}_{n=1}^{\infty} \subset \mathcal{A}_\infty \subset \mathcal{A}_p \) be defined as
\[ u^{(n)}(x, y) := \begin{cases} \left( \frac{y}{n^{1-1}} \right)^{1-1} x, & (x, y) \in \Omega_1^{(n)}; \\ \left( \frac{y}{n^{1-1}} \right)^{1-1} x, & (x, y) \in \Omega_2^{(n)}. \end{cases} \]
Clearly, each element of this sequence is in \( \mathcal{A}_\infty \).

For any \( n \in \{1, 2, \ldots \} \) we have
\[ J[u^{(n)}; m] = \int_{\Omega_{p_0, p_1}} L(x, u^{(n)}_x, u^{(n)}_y, y, m) \, dx \, dy \]
(27)
\[ = \int_{\Omega_1^{(n)}} L(x, u^{(n)}_x, u^{(n)}_y, y, m) \, dx \, dy + \int_{\Omega_2^{(n)}} L(x, u^{(n)}_x, u^{(n)}_y, y, m) \, dx \, dy. \]
The first integral in (27) approaches zero as \( n \to \infty \). As before, for the second integral we have
\[ \int_{\Omega_2^{(n)}} L(x, u^{(n)}_x, u^{(n)}_y, y, m) \, dx \, dy = \int_{\{y | p_0 < y < p_1\}} \int K(x, u^{(n)}_x, u^{(n)}_y; y, m) \, dx \, dy. \]
For each \( y \in (p_0, p_1) \), the inner integral above approaches \( \frac{1}{f(y, m)} \) as \( n \to \infty \). It follows that
\[ \lim_{n \to \infty} J[u^{(n)}; m] = \int_{\{y | p_0 < y < p_1\}} \alpha_p(y) \, dy = \alpha(p_1). \]
Since $\alpha \in \mathcal{M}_{p_0,p_1}$ and $p \geq p_1$, by definition of $\mathcal{M}_{p_0,p_1}$ we must have $\alpha(p) = \alpha(p_1)$. Therefore $\inf_{u \in \mathcal{A}_p} J[u; m] \leq \alpha(p)$ and $I(p; m) \leq \alpha(p)$ as desired. Thus the lemma is proven. \hfill \Box

6. Main Result and Summary

Lemma 1 and Lemma 2, from the previous two sections imply the following

**Theorem 1** (Main Result). Let $m \in [3p_1, \infty)$. For each $p \in [1, \infty]$, we have $I(p; m) = \alpha(p)$.

What we have shown is that there are functionals that exhibit the Lavrentiev phenomenon and have an infimum that depends continuously upon the amount of regularity required of the competing functions. In fact, let $\alpha$ be an absolutely continuous function defined on $[1, \infty]$ satisfying the following conditions:

(i) $\exists p_0 \in (1, \infty)$ such that $\alpha|_{[1,p_0]} = 0$;
(ii) $\exists p_1 \in (1, \infty)$ such that $\alpha_p|_{(p_1, \infty)} = 0$;
(iii) $\alpha_p|_{[p_0,p_1]} \geq 0$.

We have provided functionals and boundary conditions such that if we require the competing functions to be in $W^{1,p}$, then the infimum for the functional matches $\alpha(p)$. Moreover, if $\alpha_p|_{[p_0,p_1]}$ is smooth, then the Lagrangian of the functional may be made as smooth as desired, short of $C^\infty$.

There are many ways to generalize the result we have presented. In this report, we restricted our attention to a particular form for the exponent on $u$ in (4). By modifying this exponent, it is possible to construct variational problems with Dirichlet boundary conditions that are Lipschitz so that the same phenomenon is exhibited. With a different form for the exponent on $u$, one can also construct a functional with an infimum that matches an $\alpha$ with a finite number of jump discontinuities (the $\alpha$ need only satisfy conditions (i)-(iii) and be absolutely continuous away from the discontinuities).

Another possible generalization is to increase the dimensions for the domain and range of the admissible mappings. However, it is still unknown whether or not there exists a one-dimensional version of the result we have presented. In other words, we do not know if there exists an $\alpha$ satisfying conditions (i)-(iii) and a one-dimensional variational problem such that the infimum for the functional is $\alpha(p)$ when the admissible functions are required to be in $W^{1,p}$.

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**References**


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