3D-2D ASYMPTOTIC ANALYSIS
FOR MICROMAGNETIC THIN FILMS

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Abstract. Γ-convergence techniques and relaxation results of constrained energy functionals are used to identify the limiting energy as the thickness ε approaches zero of a ferromagnetic thin structure $\Omega_\varepsilon = \omega \times (-\varepsilon, \varepsilon)$, $\omega \subset \mathbb{R}^2$, whose energy is given by

$$\mathcal{E}_\varepsilon(m) = \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \left( W(m, \nabla m) + \frac{1}{2} \nabla u \cdot m \right) \, dx$$

subject to

$$\text{div}(-\nabla u + m \chi_{\Omega_\varepsilon}) = 0 \quad \text{on } \mathbb{R}^3,$$

and to the constraint

$$|m| = 1 \text{ on } \Omega_\varepsilon,$$

where $W$ is any continuous function satisfying $p$-growth assumptions with $p \geq 1$.

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1. Introduction

In recent years the understanding of thin film behavior has been helped by the mathematical asymptotic analysis of energies defined on three-dimensional domains of vanishing thickness, through the use of $\Gamma$-convergence techniques (see [ ], [ ], [ ], [ ], [ ]). The method consists in rescaling the $\varepsilon$-thin domain into a reference body of unit thickness, so that the resulting energy will be defined on a fixed domain, while the dependence on $\varepsilon$ will be explicit in the transversal derivatives which appear in the energy. The second step is then to determine the $\Gamma$-limit of the rescaled energy as the thickness $\varepsilon$ tends to 0.

In this paper and within the framework of micromagnetics, we perform the analysis described above for the energy of a ferromagnetic thin film $\Omega_\varepsilon = \omega \times (-\varepsilon, \varepsilon)$, $\omega \subset \mathbb{R}^2$, of the type

$$E_\varepsilon(m) = \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \left( W(m, \nabla m) + \frac{1}{2} \nabla \cdot m \right) dx \quad (1.1)$$

subject to

$$\text{div}(-\nabla u + m \chi_{\Omega_\varepsilon}) = 0 \quad \text{on } \mathbb{R}^3,$$

and to the constraint

$$|m| = 1 \text{ on } \Omega_\varepsilon,$$

where $W$ is any continuous function satisfying $p$-growth assumptions, with $p \geq 1$ (see Section ). Here $m : \Omega_\varepsilon \rightarrow \mathbb{R}^3$ represents the magnetization and $u$ is a scalar potential for the magnetic field $H = -\nabla u$. In particular (see Remark ) we recover the case studied by Gioia and James in [ ], where

$$W(m, \nabla m) = \gamma |\nabla m|^2 + \varphi(m),$$

and $E_\varepsilon$ represents the standard micromagnetic energy (see [ ] and [ ] for a detailed explanation of the model).

In our analysis a fundamental role is played by the characterization of the relaxation of integral functionals where the admissible fields are constrained to remain on the unit sphere. This problem has been faced in [ ], where the notion of tangential quasiconvexification $Q_T f$ of a function $f$ has been introduced (see Definition ). For $p > 1$ we show that the limit energy is

$$E(m) = \int_\omega 2Q_T \tilde{W}(m, \nabla m) + m_3^2 \, dx_1 \, dx_2 \quad \text{on } W^{1,p}(\omega) \cap \{|m| = 1\}, \quad (1.3)$$
where $\hat{W}$ is obtained by $W$ through a minimization with respect to the transversal derivatives of $m$ (see formula (1.1)). It is remarkable that in the limit the admissible fields do not depend on the direction normal to the thin film, and the magnetostatic equation (1.2) completely disappears.

So in the superlinear case ($p > 1$) we completely characterize the $\Gamma$-limit of the rescaled version of (1.1) and we derive a convergence result of minimum problems.

In the case $p = 1$, the characterization of the $\Gamma$-limit is not completely obtained, since it relies on the still open problem of finding an integral representation of relaxed functionals of the type

$$\mathcal{F}(u) = \inf \left\{ \liminf_n \int_\Omega f(u_n, \nabla u_n) \, dx, u_n \to u \text{ in } L^1(\Omega), \|\nabla u_n\|_1 \leq c, u_n \in W^{1,1}(\Omega), |u_n| = 1 \right\},$$

when $f$ is a continuous function with linear growth. The domain of $\mathcal{F}$ will be a subset of $BV(\Omega, S^2)$, the set of functions of bounded variations with values on the unit sphere $S^2$, containing properly $W^{1,1}(\Omega, S^2)$. The characterization of such domain and an integral representation of $\mathcal{F}(u)$, explicating the dependence also on the singular part of $Du$ with respect to the Lebesgue measure, is a problem on which the authors are working. As a partial result, in this paper, we characterize $\mathcal{F}$ on $W^{1,1}(\Omega, S^2)$ (see Theorem 2). This allows us to show that, in the case $p = 1$, $E(m)$ still has the same expression (1.3) on $W^{1,1}(\omega) \cap \{|m| = 1\}$.

The outline of the paper is the following. In Section 2 we recall the definitions and the main properties of relaxation and $\Gamma$-convergence. In Section 3 we state the main result concerning relaxation of constrained integral functionals obtained in [1], and in Theorem 4 we prove some extensions of this result to the linear case. Finally, Section 5 is devoted to the characterization of the $\Gamma$-limit of the rescaled version of (1.1) both in the superlinear and in the linear case (see Theorems 6 and 7).

2. Relaxation and $\Gamma$-convergence

Let $(X, d)$ be a metric space. We first recall the notion of relaxed functional. Let $F : X \to [0, +\infty]$. Then the relaxed functional $\overline{F}$ of $F$, or relaxation of $F$, is the greatest $d$-lower semicontinuous functional less than or equal to $F$. We can characterize $\overline{F}$ as follows:

$$\overline{F}(u) = \inf \{ \liminf_j F(u_j) : u_j \to u \text{ in } X \}.$$
A family $(F_\varepsilon)_{\varepsilon > 0}$ of functionals $F_\varepsilon : X \to [0, +\infty]$ is said to $\Gamma$-converge to a functional $F : X \to [0, +\infty]$ at $u \in X$, and we write $F(u) = \Gamma\lim_{\varepsilon \to 0^+} F_\varepsilon(u)$, if for every sequence $(\varepsilon_j)$ of positive numbers decreasing to 0 the following two conditions hold:
(i) (lower semicontinuity inequality) for all sequences $(u_j)$ converging to $u$ in $X$ we have $F(u) \leq \lim \inf_j F_{\varepsilon_j}(u_j)$;
(ii) (existence of a recovery sequence) there exists a sequence $(u_j)$ converging to $u$ in $X$ such that $F(u) \geq \lim \sup_j F_{\varepsilon_j}(u_j)$.

We say that $F_\varepsilon$ $\Gamma$-converges to $F$ if $F(u) = \Gamma\lim_{\varepsilon \to 0^+} F_\varepsilon(u)$ at all points $u \in X$ and that $F$ is the $\Gamma$-limit of $F_\varepsilon$. If we define the lower and upper $\Gamma$-limits by

$$F''(u) = \Gamma\lim_{\varepsilon \to 0^+} \sup F_\varepsilon(u) = \inf \{ \lim \sup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \to u \text{ in } X \},$$

$$F'(u) = \Gamma\lim_{\varepsilon \to 0^+} \inf F_\varepsilon(u) = \inf \{ \lim \inf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \to u \text{ in } X \},$$

respectively, then the conditions (i) and (ii) are equivalent to $F'(u) = F''(u) = F(u)$. Note that the functions $F'$ and $F''$ are lower semicontinuous. Moreover if $F_\varepsilon \equiv F$, for every $\varepsilon > 0$, then $F' = F'' = F$, the relaxation of $F$.

A fundamental result concerning the notion of $\Gamma$-convergence is the following theorem.

**Theorem 2.1.** Let $F = \Gamma\lim_{\varepsilon \to 0^+} F_\varepsilon$, and let $K \subset X$ be a compact set such that $\inf_X F_\varepsilon = \inf_K F_\varepsilon$ for all $\varepsilon$. Then $F$ attains its minimum on $X$ and

$$\min_X F = \lim_{\varepsilon \to 0^+} \inf_X F_\varepsilon.$$  \hfill (2.1)

Moreover, if $(u_j)$ is a converging sequence such that $\lim_j F_{\varepsilon_j}(u_j) = \lim_j \inf_X F_{\varepsilon_j}$ then its limit is a minimum point for $F$.

We refer to [ ] for an exposition of the main properties of $\Gamma$-convergence (see also [ ]).

3. Relaxation of constrained energy functionals

Let $\Omega$ be a bounded, open set of $\mathbb{R}^N$, and let $f : \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a continuous function.

The quasiconvex envelope $Q f$ of $f$ is defined by

$$Q f(u, \xi) = \inf \left\{ \int_{(0,1)^N} f(u, \xi + \nabla \varphi(x)) \, dx : \varphi \in W^{1,\infty}_0((0,1)^N, \mathbb{R}^d) \right\}$$  \hfill (3.1)
(see |] and |]). It has been shown by Dacorogna (see []) that, if f satisfies
\[ 0 \leq f(u, \xi) \leq C(1 + |\xi|^p), \quad \forall (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}, \]
for some \( C > 0, p > 1 \), then the relaxed energy
\[ \mathcal{F}(u) = \inf \left\{ \liminf_n \int_{\Omega} f(u_n, \nabla u_n) \, dx : u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^d) \right\} \]
is given by
\[ \mathcal{F}(u) = \int_{\Omega} Qf(u, \nabla u) \, dx. \]
The integral representation of the relaxed energy when the admissible fields are con-
strained to remain on a \( C^1 \) manifold \( \mathcal{M} \subset \mathbb{R}^d \), has been studied in [].

Let us consider the case \( \mathcal{M} = S^{d-1} \), the unit sphere in \( \mathbb{R}^d \). If \( u \in S^{d-1} \), we denote
by \( T_u(S^{d-1}) \) the tangent space to \( S^{d-1} \) at \( u \). Recall that \( T_u(S^{d-1}) = u^\perp \), the linear
hyperplane orthogonal to \( u \).

The following definition was introduced in [], when \( f \) does not depend on \( u \).

**Definition 3.1.** Let \( u \in S^{d-1} \) and \( \xi \in [T_u(S^{d-1})]^N \). The tangential qua-
si-convexification of \( f \) at \((u, \xi)\) is defined by
\[ Q_{T, f}^{N, d}(u, \xi) = \inf \left\{ \int_{(0,1)^N} f(u, \xi + \nabla \varphi(x)) \, dx : \varphi \in W^{1,\infty}_0((0,1)^N, T_u(S^{d-1})) \right\}. \tag{3.2} \]
We say that \( f \) is a tangential quasi-convex function if \( f(u, \xi) = Q_{T, f}^{N, d}(u, \xi) \), for any
\( u \in S^{d-1} \) and \( \xi \in [T_u(S^{d-1})]^N \).

Setting
\[ \overline{f}(u, \xi) = \begin{cases} \min\{|u|, 1\} f \left( \frac{u}{|u|}, \left( I_{d \times d} - \frac{u \otimes u}{|u|^2} \right) \xi \right) & u \neq 0 \\ 0 & u = 0, \end{cases} \tag{3.3} \]
we can prove, as in Proposition 2.2 of [], that, for \( u \in S^{d-1} \) and \( \xi \in [T_u(S^{d-1})]^N \),
\[ Q_{T, f}^{N, d}(u, \xi) = Q_{\overline{f}}(u, \xi). \tag{3.4} \]

Let us note that, if \( u \in S^{d-1} \), then \((I_{d \times d} - u \otimes u)\xi = (P_u \xi^1, \ldots, P_u \xi^N)\), for any
\( \xi = (\xi^1, \ldots, \xi^N) \in \mathbb{R}^{d \times N} \), where \( P_u \) is the orthogonal projection of \( \mathbb{R}^d \) onto the tangent
space \( T_u(S^{d-1}) \).
For any $u \in W^{1,p}(\Omega, S^{d-1})$ define the relaxed energy

$$
\mathcal{F}_T(u) = \inf \left\{ \liminf_n \int_\Omega f(u_n, \nabla u_n) \, dx : u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^d), \, u_n \in S^{d-1} \text{ a.e. in } \Omega \right\}.
$$

The following result is a slight generalization of Theorem 3.1 in [ ]. We omit the proof since it does not require any relevant improvement of the argument used in [ ].

**Theorem 3.2.** Let $f : \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a continuous function such that

$$
0 \leq f(u, \xi) \leq C(1 + |\xi|^p), \quad \forall (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N},
$$

for some $C > 0$, $p \geq 1$. Then for every $u \in W^{1,p}(\Omega, S^{d-1})$

$$
\mathcal{F}_T(u) = \int_\Omega Q_T^{N,d} f(u, \nabla u) \, dx.
$$

**Remark 3.3.** Let $f$ satisfy (3.5) and the additional coercivity assumption

$$
C_1|\xi|^p \leq f(u, \xi), \quad \forall (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N},
$$

for some constant $C_1 > 0$, and define the functional

$$
F(u) = \begin{cases}
\int_\Omega f(u, \nabla u) \, dx & \text{if } u \in W^{1,p}(\Omega, S^{d-1}) \\
+\infty & \text{if } u \in L^1(\Omega, \mathbb{R}^d) \setminus W^{1,p}(\Omega, S^{d-1}).
\end{cases}
$$

Then, for $p > 1$, as straightforward consequence of the previous result, we have that the relaxation of $F$ with respect to the $L^1$-metric

$$
\overline{F}(u) = \inf \left\{ \liminf_n F(u_n) : u_n \rightharpoonup u \text{ in } L^1(\Omega, \mathbb{R}^d) \right\},
$$

has the following integral representation

$$
\overline{F}(u) = \begin{cases}
\int_\Omega Q_T^{N,d} f(u, \nabla u) \, dx & \text{if } u \in W^{1,p}(\Omega, S^{d-1}) \\
+\infty & \text{if } u \in L^1(\Omega, \mathbb{R}^d) \setminus W^{1,p}(\Omega, S^{d-1}).
\end{cases}
$$

If $p = 1$ the domain of $\overline{F}$ will be a subset of $BV(\Omega, S^{d-1})$, the set of functions of bounded variations with values on $S^{d-1}$, containing properly $W^{1,1}(\Omega, S^{d-1})$. The characterization of such domain and an integral representation of $\overline{F}(u)$, expliciting the dependence also on the singular part of $Du$ with respect to the Lebesgue measure, is still an open problem on which the authors are working.

As a partial result, the following theorem characterizes $\overline{F}$ on $W^{1,1}(\Omega, S^{d-1})$ and it is based on the lower semicontinuity results of [ ].
Theorem 3.4. Let $f$ be a continuous function satisfying (3.5) and (3.6), with $p = 1$, and let $\overline{f}$, given in (3.3), satisfy the following hypothesis

(i) for all $u_0 \in \mathbb{R}^d$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|u - u_0| < \delta$ implies that

$$f(u, \xi) - f(u_0, \xi) \geq -\varepsilon(1 + |\xi|).$$

Then

$$F(u) = \int_{\Omega} Q_T^{N,d} f(u, \nabla u) \, dx \quad \text{on } W^{1,1}(\Omega, S^{d-1}).$$

Proof. From Theorem 3.2 it follows that

$$\overline{F}(u) \leq \int_{\Omega} Q_T^{N,d} f(u, \nabla u) \, dx \quad \text{on } W^{1,1}(\Omega, S^{d-1}).$$

The opposite inequality is a consequence of (3.4). Indeed, if $u \in W^{1,1}(\Omega, S^{d-1})$ we have that $\nabla u \in [T_u(S^{d-1})]^N$ a.e. in $\Omega$, thus

$$f(u, \nabla u) = \overline{f}(u, \nabla u) \quad \text{a.e. in } \Omega,$$

which implies

$$F(u) = \int_{\Omega} \overline{f}(u, \nabla u) \, dx \quad \text{on } W^{1,1}(\Omega, S^{d-1}).$$

Then, by virtue of (3.4) and the coercivity assumption (3.6), we get the conclusion if we prove that for any $u_n, u \in W^{1,1}(\Omega, \mathbb{R}^d)$, with $u_n$ converging to $u$ strongly in $L^1(\Omega, \mathbb{R}^d)$ and $u_n$ bounded in $W^{1,1}(\Omega, \mathbb{R}^d)$,

$$\liminf_n \int_{\Omega} \overline{f}(u_n, \nabla u_n) \, dx \geq \int_{\Omega} Q\overline{f}(u, \nabla u) \, dx.$$  

This can be done following the line of the proof of Theorem 2.1 in [ ], up to slight modifications. Note that the hypothesis (i) corresponds to condition (H4) in [ ] and (3.5) yields

$$0 \leq \overline{f}(u, \xi) \leq C(1 + |\xi|), \quad (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}.$$  

The hypothesis of coercitivity on $\overline{f}$, required in Theorem 2.1 of [ ], is not needed by the boundedness of $u_n$ in $W^{1,1}(\Omega, \mathbb{R}^d)$. 

$\Box$
Remark 3.5. Note that, even if the function $f$ does not satisfy the coercitivity condition (3.6), the same conclusion of Theorem 3.4 still holds for the functional

$$\tilde{F}(u) = \inf \left\{ \liminf_n F(u_n) : u_n \to u \text{ in } L^1(\Omega, \mathbb{R}^d), \ (u_n) \text{ bounded in } W^{1,1}(\Omega, \mathbb{R}^d) \right\}.$$ 

Remark 3.6. It is easy to prove that, if $f(u, \xi)$ is Lipschitz in $\xi$ uniformly with respect to $u$, the hypothesis (i) on $\tilde{f}$ in Theorem 3.4 is implied by the same hypothesis on $f$.

4. Limit of micromagnetic energies on thin films

For $\varepsilon > 0$ let $\Omega_{\varepsilon}$ be a thin three-dimensional domain of the form $\Omega_{\varepsilon} = \omega \times (-\varepsilon, \varepsilon)$, with $\omega$ a bounded open set of $\mathbb{R}^2$, and denote $\Omega := \Omega_1$. Let $p \geq 1$ and let $W : \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \to \mathbb{R}$ be a continuos function such that, for any $(m, \xi) \in \mathbb{R}^3 \times \mathbb{R}^{3 \times 3}$,

$$\frac{1}{C} |\xi|^p \leq W(m, \xi) \leq C(1 + |\xi|^p),$$

(4.1)

for some constant $C > 0$.

Define, for any $\overline{m} \in L^1(\Omega_{\varepsilon}, \mathbb{R}^3)$,

$$\mathcal{E}_\varepsilon(\overline{m}) = \begin{cases} \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} \left( W(\overline{m}, \nabla \overline{m}) + \frac{1}{2} \nabla u \cdot \overline{m} \right) \, dx & \text{if } \overline{m} \in W^{1,p}(\Omega_{\varepsilon}, \mathcal{S}^2) \\ +\infty & \text{otherwise,} \end{cases}$$

(4.2)

where $\overline{u}$ is related to $\overline{m}$ by the equation

$$\text{div}(-\nabla \overline{u} + \overline{m}) = 0 \quad \text{on } \mathbb{R}^3,$$

(4.3)

with $\overline{m}$ extended by 0 outside $\Omega_{\varepsilon}$.

Through the change of variables

$$m(x_\alpha, x_3) = \overline{m}(x_\alpha, \varepsilon x_3), \quad u(x_\alpha, x_3) = \overline{u}(x_\alpha, \varepsilon x_3), \quad x_\alpha = (x_1, x_2) \in \omega, \ x_3 \in (-1, 1),$$

we rescale the functional (4.2) as

$$E_\varepsilon(m) = \begin{cases} \int_{\Omega} \left( W(m, \nabla m, \frac{1}{\varepsilon} \nabla_3 m) + \frac{1}{2} \int_{\mathbb{R}^3} \left( \nabla u \cdot m_\alpha + \frac{1}{\varepsilon} m_3 \nabla_3 u \right) \, dx \right) & \text{if } m \in W^{1,p}(\Omega, \mathcal{S}^2) \\ +\infty & \text{otherwise,} \end{cases}$$

(4.4)
subjected to the constraint
\[
\text{div}(-\nabla_x u + m_\alpha) + \frac{1}{\varepsilon} \nabla_3(-\frac{1}{\varepsilon} \nabla_3 u + m_3) = 0 \quad \text{on } \mathbb{R}^3, \tag{4.5}
\]
where we have used the notation
\[
m_\alpha = (m_1, m_2), \quad \nabla_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2, 3, \quad \nabla_\alpha = (\nabla_1, \nabla_2).
\]

We now proceed to clarify the meaning of the magnetostatic equation (4.5). Consider the following variational principle
\[
\min_{u \in V} \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_\alpha u - m_\alpha|^2 + \frac{1}{\varepsilon^2} |\nabla_3 u - m_3|^2 \, dx, \tag{4.6}
\]
where
\[
V = \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^3) : \nabla v \in L^2(\mathbb{R}^3, \mathbb{R}^3), \int_B v \, dx = 0 \right\},
\]
and \(B\) is a fixed ball containing \(\Omega\). \(V\) is a Hilbert space with inner product
\[
(u, v)_V = \int_{\mathbb{R}^3} \nabla_\alpha u \cdot \nabla_\alpha v + \frac{1}{\varepsilon^2} \nabla_3 u \nabla_3 v \, dx.
\]
The direct method of the calculus of variations yields a unique minimizer of (4.6) in \(V\), satisfying the Euler-Lagrange equation
\[
\int_{\mathbb{R}^3} (\nabla_\alpha u - m_\alpha) \cdot \nabla_\alpha v + \frac{1}{\varepsilon} \left( \frac{1}{\varepsilon} \nabla_3 u - m_3 \right) \nabla_3 v \, dx = 0 \quad \forall v \in V, \tag{4.7}
\]
that is the weak form of (4.5). Setting \(v = u\) in (4.7), and taking into account that \(m\) vanishes outside \(\Omega\), we obtain
\[
\int_{\Omega} \nabla_\alpha u \cdot m_\alpha + \frac{1}{\varepsilon} m_3 \nabla_3 u \, dx = \int_{\mathbb{R}^3} |\nabla_\alpha u|^2 + \frac{1}{\varepsilon^2} |\nabla_3 u|^2 \, dx. \tag{4.8}
\]
Note that the left-hand side of this expression is twice the magnetostatic energy given by the second integral in (4.4).

The following proposition is due to G. Gioia and R.D. James (Proposition 4.1 of [ ]).

**Proposition 4.1.** Suppose \(m_\varepsilon \rightarrow m\) in \(L^2(\mathbb{R}^3, \mathbb{R}^3)\), \(m_\varepsilon = 0\) on \(\mathbb{R}^3 \setminus \Omega\), and let \(u_\varepsilon\) the minimizer of (4.6) with \(m = m_\varepsilon\). Then
\[
\nabla u_\varepsilon \rightarrow 0, \quad \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \rightarrow m_3 \quad \text{in } L^2(\mathbb{R}^3).
\]

We now state the thin-film approximation result in the superlinear case.
Theorem 4.2. If $p > 1$, then $E_{\varepsilon}$ $\Gamma$-converges with respect to the $L^1$-strong topology to the functional $E : L^1(\Omega, \mathbb{R}^3) \to [0, +\infty]$ defined as

$$
E(m) = \begin{cases} 
2 \int_\omega Q_T^{3,3} \tilde{W}(m, \nabla m) \, dx_{\alpha} + \int_\omega |m_3|^2 \, dx_{\alpha} & \text{if } m \in W^{1,p}(\omega, \mathbb{S}^2) \\
+\infty & \text{otherwise},
\end{cases}
$$

(4.9)

where $\tilde{W} : \mathbb{R}^3 \times \mathbb{R}^{3 \times 2} \to \mathbb{R}$ is given by

$$
\tilde{W}(m, \zeta) := \inf_{z \in m^\perp} W(m, \zeta, z).
$$

(4.10)

Remark 4.3. It is easy to check, by the continuity and the growth assumptions (4.1) on $W$, that $\tilde{W}$ satisfies the hypotheses of Theorem 3.2.

Proof of Theorem 4.2. As usual we divide the proof in two steps dealing with the $\Gamma$-lim inf and $\Gamma$-lim sup inequality, separately.

Step 1. $\Gamma$-lim inf inequality.

Let $\varepsilon_h \searrow 0$ and let $m_h \to m$ in $L^1(\Omega, \mathbb{R}^3)$ be such that $\liminf_h E_{\varepsilon_h}(m_h) < +\infty$. Then, up to a subsequence, by the coercivity assumption (4.1), we have that $m_h \in W^{1,p}(\Omega, \mathbb{S}^2)$ and

$$
\sup_h \int_\Omega |\nabla m_h|^p + \frac{1}{\varepsilon_h} |\nabla_3 m|^p \, dx < +\infty.
$$

Thus $m_h$ converges to $m$ weakly in $W^{1,p}(\Omega, \mathbb{R}^3)$, strongly in $L^q(\Omega, \mathbb{R}^3)$, for every $q < +\infty$, $m \in W^{1,p}(\Omega, \mathbb{S}^2)$, and $\nabla_3 m = 0$, that is $m$ does not depend on the transverse direction $x_3$.

Since $\nabla_3 m_h \in m_h^\perp$, we have

$$
E_{\varepsilon_h}(m_h) \geq \int_\Omega \tilde{W}(m_h, \nabla_\alpha m_h) \, dx + \frac{1}{2} \int_\Omega |\nabla_\alpha u_h|^2 + \frac{1}{\varepsilon_h^2} |\nabla_3 u_h|^2 \, dx,
$$

where $u_h$ is the solution of (4.5) corresponding to $m_h$. Then Theorem 3.2 and Proposition 4.1 yield

$$
\liminf_h E_{\varepsilon_h}(m_h) \geq \int_\Omega Q_T^{3,3} \tilde{W}(m, \nabla_\alpha m) \, dx + \frac{1}{2} \int_\Omega |m_3|^2 \, dx.
$$

A straightforward application of Fubini’s Theorem in the definition (3.2) shows that $Q_T^{3,3} \tilde{W}(m, \zeta) \geq Q_T^{2,3} \tilde{W}(m, \zeta)$, $(m, \zeta) \in \mathbb{R}^3 \times \mathbb{R}^{3 \times 2}$. We conclude, by the arbitrariness of the sequence $(\varepsilon_h)$, that

$$
\Gamma \text{-lim inf}_{\varepsilon \to 0} E_{\varepsilon}(m) \geq E(m),
$$
for any $m \in L^1(\Omega, \mathbb{R}^3)$.

**Step 2.** $\Gamma$-lim sup inequality.

We want to show that, for every $m \in W^{1,p}(\omega, S^2)$,

$$
\Gamma \text{-lim sup } E_\varepsilon(m) \leq 2 \int_\omega \dot{W}(m, \nabla m) \, dx_\alpha + \int_\omega |m_3|^2 \, dx_\alpha,
$$

so that, by the lower semicontinuity of $\Gamma$-lim sup and by Theorem 3.2 and Remark 3.3, we can conclude that

$$
\Gamma \text{-lim sup } E_\varepsilon(m) \leq E(m),
$$

for any $m \in L^1(\Omega, \mathbb{R}^3)$, and then by Step 1 we get the thesis.

Let $m \in W^{1,p}(\omega, S^2)$. For any $\eta > 0$, a measurability selection criterion (see [ ] for example) allows us to find a measurable function $\tilde{z} : \omega \to \mathbb{R}$ such that $\tilde{z}(x) \in m(x) \perp$ a.e. in $\omega$ and

$$
\int_\omega \dot{W}(m, \nabla m) \, dx_\alpha \geq \int_\omega W(m, \nabla m, \tilde{z}) \, dx_\alpha - \eta.
$$

The growth assumptions on $W$ and $\dot{W}$ ensure that $\tilde{z} \in L^p(\omega, \mathbb{R}^3)$. Consider a sequence $\tilde{z}_n \in C^\infty_c(\omega, \mathbb{R}^3)$ which approximates $\tilde{z}$ strongly in $L^p(\omega, \mathbb{R}^3)$, and set

$$
z_n = (I - m \otimes m)\tilde{z}_n,
$$

that is the projection of $\tilde{z}_n$ on $m \perp$. We have that $z_n \in W^{1,p}(\omega, \mathbb{R}^3) \cap L^\infty(\omega, \mathbb{R}^3)$ and $z_n$ still converges to $\tilde{z}$ strongly in $L^p(\omega, \mathbb{R}^3)$. Then, we can find $n(\eta)$ such that

$$
\int_\omega \dot{W}(m, \nabla m) \, dx_\alpha \geq \int_\omega W(m, \nabla m, z_n(\eta)) \, dx_\alpha - 2\eta.
$$

Set $z = z_n(\eta)$ and define, for $\varepsilon > 0$,

$$
\bar{m}_\varepsilon(x) = m(x_\alpha) + \varepsilon x_3 z(x_\alpha), \quad m_\varepsilon(x) = \frac{\bar{m}_\varepsilon(x)}{|\bar{m}_\varepsilon(x)|}, \quad x \in \Omega.
$$

Since $z \in W^{1,p}(\omega, \mathbb{R}^3) \cap L^\infty(\omega, \mathbb{R}^3)$, then, for $\varepsilon$ sufficiently small, $m_\varepsilon$ is well defined and belongs to $W^{1,p}(\Omega, S^2)$; moreover, both $\bar{m}_\varepsilon$ and $m_\varepsilon$ converge to $m$ strongly in $L^\infty(\Omega, \mathbb{R}^3)$. An easy computation shows that

$$
\nabla m_\varepsilon = \left( \frac{I - \bar{m}_\varepsilon \otimes \bar{m}_\varepsilon}{|\bar{m}_\varepsilon|^3} \right) \nabla \bar{m}_\varepsilon.
$$
In particular, as \( \nabla m_\varepsilon = (\nabla \alpha m + \varepsilon x_3 \nabla \alpha z, \varepsilon z) \), we have
\[
\nabla \alpha m_\varepsilon = \left( \frac{I}{|m_\varepsilon|} - \frac{m_\varepsilon \otimes m_\varepsilon}{|m_\varepsilon|^3} \right) (\nabla \alpha m + \varepsilon x_3 \nabla \alpha z), \quad \nabla_3 m_\varepsilon = \left( \frac{I}{|m_\varepsilon|} - \frac{m_\varepsilon \otimes m_\varepsilon}{|m_\varepsilon|^3} \right) \varepsilon z
\]
and, letting \( \varepsilon \) tend to 0,
\[
\nabla \alpha m_\varepsilon \to (I - m \otimes m) \nabla \alpha m = \nabla \alpha m, \quad \frac{1}{\varepsilon} \nabla_3 m_\varepsilon \to (I - m \otimes m) z = z,
\]
strongly in \( L^p(\Omega, \mathbb{R}^3) \).

Thus, by (4.1) and by Proposition 4.1, we get
\[
\lim_{\varepsilon \to 0} E_\varepsilon(m_\varepsilon) = 2 \int_\omega W(m, \nabla m, z) \, dx + \int_\omega |m_3|^2 \, dx
\leq 2 \int_\omega \hat{W}(m, \nabla m) \, dx + \int_\omega |m_3|^2 \, dx + 2\eta.
\]
The inequality (4.11) easily follows by the definition of \( \Gamma \)-lim sup and the arbitrariness of \( \eta > 0 \).

**Remark 4.4.** The previous result can be easily extended to the case of an applied field \( h \) by substituting \( E_\varepsilon(m) - \int_\Omega h \cdot m \, dx \) for \( E_\varepsilon(m) \) (with \( h \in L^1(\Omega, \mathbb{R}^3) \)), since \( \Gamma \)-convergence is stable under continuous perturbations.

**Remark 4.5.** If \( W \) is also a tangential quasiconvex function, it can be easily proved the existence of a minimizer of \( E_\varepsilon(m) - \int_\Omega h \cdot m \, dx \) in \( W^{1,p}(\Omega, \mathbb{S}^2) \), by applying the direct method of the calculus of variations.

As a straightforward consequence of Theorem 2.1, we derive from Theorem 4.2 the following result on the convergence of minimum problems.

**Corollary 4.6.** Let \( W \) be a tangential quasiconvex function, \( h \in L^1(\Omega, \mathbb{R}^3) \), and let \( m_\varepsilon \in W^{1,p}(\Omega, \mathbb{S}^2) \) be a minimizer of \( E_\varepsilon(m) - \int_\Omega h \cdot m \, dx \). Then, for every sequence \( (\varepsilon_h) \) tending to 0 there exists a subsequence (not relabelled) \( (m_{\varepsilon_h}) \) converging weakly in \( W^{1,p}(\Omega, \mathbb{R}^3) \) to a function \( m \in W^{1,p}(\omega, \mathbb{S}^2) \) such that \( m \) is a minimizer of
\[
\min \left\{ 2 \int_\omega \hat{W}(m, \nabla m) \, dx\alpha + \int_\omega |m_3|^2 \, dx\alpha - \int_\Omega h \cdot m \, dx, \ m \in W^{1,p}(\omega, \mathbb{S}^2) \right\}.
\]
Remark 4.7. If \( W(m, \xi) = \gamma|\xi|^2 + \varphi(m) \), that is \( E_{\varepsilon} \) represents the standard micromagnetic energy, we have that
\[
Q_T^{2,3} \hat{W}(m, \xi) = \gamma|\xi|^2 + \varphi(m),
\]
and we recover the convergence result of minimum problems obtained in [ ].

If \( p = 1 \), as a consequence of Theorem 3.4 we obtain the following \( \Gamma \)-convergence result.

Theorem 4.8. Let \( p = 1 \) and let \( W \) satisfy the additional hypotheses:
(a) \( W(m, \xi) \) is Lipschitz in \( \xi \) uniformly with respect to \( m \);
(b) for all \( m_0 \in \mathbb{R}^d \) and for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |m - m_0| < \delta \) implies that
\[
W(m, \xi) - W(m_0, \xi) \geq -\varepsilon(1 + |\xi|).
\]

Then
\[
\Gamma(L^1) \lim_{\varepsilon \to 0} E_{\varepsilon}(m) = 2 \int_\omega Q_T^{2,3} \hat{W}(m, \nabla m) dx + \int_\omega |m_3|^2 dx \quad \text{on } W^{1,1}(\omega, S^2).
\]

Proof. It can be easily proved that hypotheses (a) and (b) are inherited by \( \hat{W} \). Then we can proceed as in the proof of Theorem 4.2, by taking into account Theorem 3.4 and Remarks 3.5, 3.6.

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References


