(1) Let \( n > 0 \). Prove that for any \( n \times n \) matrix \( A \), the following statements are equivalent:
(a) \( AB = BA \) for all \( n \times n \) matrices \( B \).
(b) \( AB = BA \) for all invertible \( n \times n \) matrices \( B \).

Claim: \( AB = BA \) for all \( B \) if and only if \( A = r1_{n \times n} \) for some real \( r \).
Proof: Clearly if \( A = r1 \) then \( AB = BA = rB \). For converse, let \( AB = BA \) for all \( B \). Let \( D_{ij} \) be the \( n \times n \) matrix with a 1 in the \( ij \) place and zeroes elsewhere. Then easily \( AD_{ij} \) is the matrix whose \( j \) column is the \( i \) column of \( A \), with zeroes elsewhere. Similarly \( D_{ij}A \) is the matrix whose \( i \) row is the \( j \) row of \( A \), with zeroes elsewhere. Since \( AD_{ij} = D_{ij}A \), we see that the row \( i \) and column \( j \) are zero except at the diagonal entry and that \( a_{ii} = a_{jj} \). This proves the claim.

Claim: if \( AB = BA \) for all invertible \( B \) then \( AB = BA \) for all \( B \).
Proof: Note that for all \( i, j \) the matrix \( 1_{n \times n} + D_{ij} \) is invertible. So \( A + AD_{ij} = D_{ij} + A \), hence \( AD_{ij} = D_{ij}A \). Now we are done by the first part.

(2) Let \( n > 0 \) and let \( A \) be an \( n \times n \) matrix. For all \( t \geq 0 \), let \( N_t \) be the nullspace of \( A^t \), where by convention \( A^0 = 1_{n \times n} \).

Prove that:
(a) \( N_t \subseteq N_{t+1} \) for all \( t \).
   If \( v \in N_t \) then \( A^t v = 0 \), so \( A^{t+1} v = A(A^t v) = A0 = 0 \) and \( v \in N_{t+1} \).
(b) The dimension of \( N_t \) (the nullity of \( A^t \)) is eventually constant, that is there is a number \( d \) such that \( \text{dim}(N_t) = d \) for all sufficiently large \( t \). \( \text{dim}(N_t) \) is an integer, is increasing and bounded above by \( n \), so is eventually constant.
(c) If \( T \) is the least \( t \) such that \( \text{dim}(N_t) = d \), then \( T \leq d \).
   \( N_t \neq N_{t+1} \) if and only if \( \text{dim}(N_t) < \text{dim}(N_{t+1}) \). If \( N_t = N_{t+1} \) then we note that
   \[ v \in N_{t+2} \implies Av \in N_{t+1} \implies Av \in N_t \implies v \in N_{t+1} \, \]
   so that \( N_{t+1} = N_{t+2} \).
   It follows that as function of \( t \) the number \( \text{dim}(N_t) \) is strictly increasing for an initial segment of \( N \), and then becomes constant. Since the eventual value is \( d \), clearly \( T \leq d \).

Proof that for all integers \( n > 0 \), \( d \) and \( T \) such that \( 0 < T \leq d \leq n \) there is a matrix \( A \) such that the nullity of \( A^t \) is \( d \) for all \( t \geq T \), and \( T \) is the least \( t \) such that the nullity of \( A^t \) is \( d \).

Rather than give the matrix \( A \) explicitly we describe where it takes each basis element in the standard basis. Note that the nullity of a \( n \times n \) matrix
$B$ is $n - \text{rank}(B)$, so what we need is a matrix $A$ such that $\text{rank}(A^t)$ stabilises at $n - d$ for $t \geq T$.

Let $A$ be the matrix which fixes $e_j$ for $1 \leq j \leq n - d$, moves $e_j$ to $e_{j+1}$ for $n - d < j < n - d + T$, and sends $e_j$ to 0 for $n - d + T \leq j \leq n$.

(3) Let $0 < m < n$. Let $A$ be $n \times m$ and let $B$ be $m \times n$. Prove that $AB$ is not invertible. Is it true in general that $BA$ is not invertible?

$AB$ is an $n \times n$ matrix. $B$ has rank at most $m$. Since $(AB)v = A(Bv)$, the column space of $AB$ is the set of $Aw$ where $w$ is in the column space of $B$. So if we fix a basis for the column space of $B$ and multiply each element from the left by $A$ we get a set which spans the column space of $AB$, hence $\text{rank}(AB) \leq \text{rank}(B) \leq m < n$ and $AB$ is not invertible.

If $A = (12)$ and $B = A^T$ then $AB = AA^T = (5)$ is invertible.

(4) For the purposes of the following question we will identify the real number $r$ with the $1 \times 1$ matrix whose only entry is $r$. In particular when we write “$M \geq 0$” when $M$ is $1 \times 1$ we mean that the entry of $M$ is non-negative.

Let $m > 1$ and let $X$ be a $1 \times m$ matrix, that is a row vector of length $m$. Prove that:

(a) $XX^T \geq 0$.

$XX^T$ is the sum of the squares of the entries in $X$.

(b) $XX^T$ has rank one if $X \neq 0$.

If $X \neq 0$ then $XX^T \neq 0$, so has nonzero rank. But the column space of $XX^T$ is contained in the column space of $X$, a space of dimension 1, so $XX^T$ has rank exactly one.

(c) $X^TX$ is symmetric.

$(X^TX) = X^T(X^T)^T = X^TX$.

(d) $A(X^TX)A^T \geq 0$ for all row vectors $A$ of length $m$.

$A(X^TX)A^T = (AX^T)(XA^T)$. Since $AX^T$ is $1 \times 1$ it is equal to its transpose so $AX^T = XA^T = (r)$ say, and now $A(X^TX)A^T \geq 0 = (r^2) \geq 0$. 