**Introduction**

$k$ indistinguishable balls, each given one of $n$ distinct colours.

$$f(n, k) = \# \text{ possible colourings.}$$

Ex. $n = k = 3$

<table>
<thead>
<tr>
<th>3R</th>
<th>2R+1B</th>
<th>2R+1W</th>
</tr>
</thead>
<tbody>
<tr>
<td>3B</td>
<td>2B+1R</td>
<td>2B+1W</td>
</tr>
<tr>
<td>3W</td>
<td>2W+1R</td>
<td>2W+1B</td>
</tr>
</tbody>
</table>

$$1R+1B+1W$$

$$f(3, 3) = 10.$$ 

Alternatively, if $x_i$ denotes the number of balls coloured $i$ then

$$x_1 + x_2 + \cdots x_n = k$$

and $f(n, k)$ is the number of non-negative integer solutions to the above equation.

Special Cases:

- $f(1, k) = 1$
- $f(n, 1) = n$
- $f(2, k) = k + 1$

General approach needed to find $f(n, k)$

More examples of recurrence relations:

**Fibonacci sequence:** 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

$$a_0 = 1, a_1 = 1 \quad \text{boundary condition}$$

$$a_n = a_{n-1} + a_{n-2}.$$ 

$a_n$ is number of rabbits at the end of $n$ periods. Each rabbit born in period $n - 2$ starts producing rabbits, one per period, when it is 2 periods old.

Simpler example: Suppose $a_1 = 1$ and

$$a_{n+1} = n a_n$$

$$\vdots$$

$$= n(n-1)(n-2) \ldots 2a_1$$

$$= n!$$
Approach 2: Generating Functions

Consider \((1 - x)^{-n} = (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)\ldots (1 + x + x^2 + \cdots)\).

What is the coefficient of \(x^k\)?

Each term is obtained by taking \(x^{t_1}\) from the first bracket, taking \(x^{t_2}\) from the second bracket, \ldots, taking \(x^{t_n}\) from the \(n\)th bracket so that \(t_1 + t_2 + \cdots + t_n = k\).

Thus this coefficient is \(f(n, k)\) and we write

\[
f(n, k) = [x^k](1 - x)^{-n} = [x^k](1 + nx + \frac{n(n+1)}{2}x^2 \cdots)
\]

Approach 3: Injective Mapping:

Put \(k\) \(X\)'s and \(n-1\) \(O\)'s in a line:

\[
XXOXOXOXOX
\]

Corresponds to \(x_1 = 2, x_2 = 1, x_3 = 1, x_4 = 0, x_5 = 1\). In general there is a 1-1 correspondence between

\[
\{\text{colourings of balls}\}
\]

and

\[
\{\text{sequences of } k \text{ \(X\)'s and } n-1 \text{ \(O\)'s}\}.
\]

Number of sequences of \(k\) \(X\)'s and \(n-1\) \(O\)'s is number of ways of choosing \(k\) positions (for the \(X\)'s) from \(n-1+k\) positions or

\[
\binom{n-1+k}{k}
\]